

Drinfel'd Twist and q -Deforming Maps for Lie Group Covariant Heisenberg Algebras

Gaetano Fiore*

Sektion Physik der Ludwig-Maximilians-Universität München

Theoretische Physik — Lehrstuhl Professor Wess

Theresienstraße 37, 80333 München

Federal Republic of Germany

Abstract

Any deformation of a Weyl or Clifford algebra can be realized through a change of generators in the undeformed algebra. q -Deformations of Weyl or Clifford algebras that were covariant under the action of a simple Lie algebra \mathfrak{g} are characterized by their being covariant under the action of the quantum group $U_h\mathfrak{g}$, $q := e^h$. We present a systematic procedure for determining all possible corresponding changes of generators, together with the corresponding realizations of the $U_h\mathfrak{g}$ -action. The intriguing relation between \mathfrak{g} -invariants and $U_h\mathfrak{g}$ -invariants suggests that these changes of generators might be employed to simplify the dynamics of some \mathfrak{g} -covariant quantum physical systems.

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e-mail: Gaetano.Fiore@physik.uni-muenchen.de

1 Introduction

Weyl and Clifford algebras (respectively denoted by $\mathcal{A}_+, \mathcal{A}_-$ in the sequel, and collectively as “Heisenberg algebras”) are at the hearth of quantum physics. One may ask if deforming them within the category of associative algebras (i.e. deforming their defining commutation relations) yields new physics, or at least may be useful to better describe certain systems in conventional quantum physics. This question can be divided into an algebraic and a representation-theoretic subquestion.

The first was addressed in the fundamental paper [1]. Essentially, it reads: is there a formal realization of the elements of the deformed algebra in terms of elements of the undeformed algebra? The answer is affirmative, but in general the realization is not explicitly known. A general result [2] regarding the Hochschild cohomology of the universal enveloping algebra associated to a nilpotent Lie group states in particular that the first and second cohomology groups of any Weyl algebra \mathcal{A}_+ are trivial. This implies [3] that any deformation \mathcal{A}_+^h (h denoting the deformation parameter) of the latter is trivial, in the sense that there exists an isomorphism of topological algebras over $\mathbf{C}[[h]]$ (a “deforming map”, in the terminology of Ref. [4]), $f : \mathcal{A}_+^h \rightarrow \mathcal{A}_+[[h]]$, reducing to the identity in the limit $h = 0$ (a concise and effective presentation of these results can be found in Sect.’s 1,2 of Ref. [5]). Practically this means that the generators $\tilde{A}^i, \tilde{A}_i^+$ of \mathcal{A}_+^h are mapped by f into power series in h with coefficients in \mathcal{A}_+ , $A^i := f(\tilde{A}^i)$, $A_i^+ = f(\tilde{A}_i^+)$, fulfilling the same deformed commutation relations and going to the corresponding generators a^i, a_i^+ of \mathcal{A}_+ in the limit $h = 0$.

The second subquestion is: do also the deformed and undeformed representation theories coincide? One can already see in some simple model that the answer is negative, but in the general case, up to our knowledge, the relation between the two is an open question.

Given any automorphism $g : \mathcal{A}_+[[h]] \rightarrow \mathcal{A}_+[[h]]$, $g = \text{id} + O(h)$, then $g \circ f$ is a new deforming map; conversely, given two deforming maps f, f' , the map $f' \circ f^{-1}$ is an algebra automorphism. Now, by the vanishing of the first cohomology group of \mathcal{A}_+ , all automorphisms of $\mathcal{A}_+[[h]]$ are ‘inner’, i.e. of the form $g(a) = \alpha a \alpha^{-1}$. Hence, all deforming maps can be obtained from one through the formula

$$f_\alpha(a) := \alpha f(a) \alpha^{-1} \quad \alpha = 1 + O(h) \in \mathcal{A}_+^h. \quad (1.1)$$

These results apply [5] in particular to so-called “ q -deformations” ($q := e^h$) of Weyl algebras which are covariant under the action of some simple Lie algebra \mathfrak{g} ;

such deformations [6, 7, 8] are matched to the deformation of $U\mathbf{g}$ into the quantum group $U_h\mathbf{g}$ [10], in the sense that for all q the deformed algebras are in fact $U_h\mathbf{g}$ -module algebras^a: the commutation relations among $\tilde{A}^i, \tilde{A}_i^+$ are compatible with the quantum group action $\tilde{\triangleright}_h : U_h\mathbf{g} \times \mathcal{A}_+^h \rightarrow \mathcal{A}_+^h$ as the commutation relations among a^i, a_i^+ were compatible with the classical action $\triangleright : U\mathbf{g} \times \mathcal{A}_+ \rightarrow \mathcal{A}_+$.

Also quantum groups admit algebra isomorphisms $\varphi_h : U_h\mathbf{g} \rightarrow U\mathbf{g}[[h]]$ [11]^b. In spite of the existence of the algebra isomorphisms f, φ_h , the $U_h\mathbf{g}$ -module algebra structure $(U_h\mathbf{g}, \mathcal{A}_+^h, \tilde{\triangleright}_h)$ is a *non-trivial* deformation of the one $(U\mathbf{g}, \mathcal{A}_+, \triangleright)$, *i.e.* for no φ_h, f the equality $f \circ \tilde{\triangleright}_h = \triangleright \circ (\varphi_h \times f)$ holds^c. This is because $U_h\mathbf{g}$ itself as a Hopf algebra is a *non-trivial* deformation of $U\mathbf{g}$, in other words all φ_h 's are algebra but not coalgebra (and therefore not Hopf algebra) isomorphisms (this is related to the non-triviality of the Gerstenhaber-Schack cohomology [12]).

Given f we define \triangleright_h as the map making the following diagram commutative:

$$\begin{array}{ccc} U_h\mathbf{g} \times \mathcal{A}_\pm^h & \xrightarrow{\tilde{\triangleright}_h} & \mathcal{A}_\pm^h \\ \downarrow \text{id} \times f & & \downarrow f \\ U_h\mathbf{g} \times \mathcal{A}_\pm[[h]] & \xrightarrow{\triangleright_h} & \mathcal{A}_\pm[[h]] \end{array} \quad (1.2)$$

(in other words $\triangleright_h := f \circ \tilde{\triangleright}_h \circ (\text{id} \otimes f^{-1})$ will realize $\tilde{\triangleright}_h$ on $\mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]]$). In this work we give a systematic procedure to construct all pairs (f, \triangleright_h) . It is based on the properties of the ‘‘Drinfel’d twist’’ [11]. We first construct \triangleright_h , then f .

One particular \triangleright_h can be naturally constructed in a ‘adjoint-like’ way (Sect. 3). To determine a corresponding f it is sufficient to identify in $\mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]]$ appropriate realizations A^i, A_i^+ of $\tilde{A}^i, \tilde{A}_i^+$. With this aim in mind, we first show [formula (3.6)] how to construct candidates A^i, A_i^+ having the same transformation properties under \triangleright_h as the generators $\tilde{A}^i, \tilde{A}_i^+$ of $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$ under $\tilde{\triangleright}_h$; they are determined up to multiplication by some \mathbf{g} -invariants. By requiring that the A^i, A_i^+ also have the same commutation rules as the $\tilde{A}^i, \tilde{A}_i^+$ (Sect. 5) we find constraints (5.0.1-5.0.3) on these \mathbf{g} -invariants. Requiring that the $*$ -structures of \mathcal{A}_\pm realize the $*$ -structures of \mathcal{A}_\pm^h yields further constraints (Sect. 6). The subalgebras $\mathcal{A}_\pm^{inv}[[h]], \mathcal{A}_\pm^{h, inv}[[h]]$ of

^aThey should not be confused with the celebrated Biedenharn-Macfarlane-Hayashi q -oscillator (super)algebras [9], whose generators are *not* $U_h\mathbf{g}$ -covariant (in spite of the fact that they are usually used to construct a generalized Jordan-Schwinger realization of $U_h\mathbf{g}$).

^b The existence of the latter and their being defined up to inner automorphisms of $U\mathbf{g}[[h]]$ again is a consequence of the triviality of the first and second Hochschild cohomology groups of $U\mathbf{g}$.

^c By \triangleright we mean here actually its linear extension to $\triangleright : U\mathbf{g}[[h]] \times \mathcal{A}_{+, \mathbf{g}, \rho}[[h]] \rightarrow \mathcal{A}_{+, \mathbf{g}, \rho}[[h]]$, where both the domain and codomain have to be understood completed in the h -adic topology.

$\mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]]$ that are invariant respectively under $\triangleright, \triangleright_h$ coincide (Sect. 4), but invariants in the form of polynomials in A^i, A_j^+ are highly non-polynomial (analytic) functions of the classical invariants and hence of a^i, a_j^+ , and conversely; we explicitly find these functions. Solving the system of equations (5.0.1-5.0.3) depends on the particular \mathbf{g} and on the particular $U\mathbf{g}$ -representation ρ to which the generators a_i^+ belong (the a^i necessarily belong to the contragradient ρ^\vee of ρ). We shall denote by $\mathcal{A}_{\pm, \mathbf{g}, \rho}$ the corresponding Heisenberg algebra and by $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$ its q -deformed version. We solve (Sect. 5) the above equations for the well-known $\mathcal{A}_{\pm, sl(N), \rho_d}^h, \mathcal{A}_{+, so(N), \rho_d}^h$, (ρ_d will denote the defining representations of either \mathbf{g}). Finally, in Sect. 7 we extend the previous results to all other isomorphisms (1.1) while giving an outlook of the whole construction, we make some remarks on the representation theory, and we draw the conclusions. This serves also to better clarify the motivations for the present work.

In Ref. [13] we started the program just sketched by sticking to the cases of triangular deformations of the Hopf algebra $U\mathbf{g}$ (in the present setting it is recovered by postulating a trivial coassociator) and of the q -deformation $\mathcal{A}_{\pm, sl(2), \rho_d}^h$.

Examples of q -deforming maps for Heisenberg algebras were explicitly constructed “by hand” in past works [14, 15, 5, 16]^d. Of course these deforming maps are related to ours by some automorphism (1.1); in Sect. 5.1 we determine the latter for $\mathcal{A}_{+, sl(2), \rho_d}^h$ and the deforming map found in Ref. [15]. We underline that our construction is based instead on universal objects characterizing the quantum symmetry algebra. This allows the application of our method *e.g.* to the physically relevant case that ρ be the direct sum of an arbitrary number of copies of ρ_d .

2 Preliminaries and notation

Some general remarks before starting. Although we will always denote the generators of the Heisenberg algebras by $a^i, a_i^+, A^i, A_i^+, \dots$, only the choice of a $*$ -structure may give them the meaning of creators/annihilators, or coordinates/derivatives, etc. (see *e.g.* Sect. 6). Given an algebra B , $B[[h]]$ will denote the algebra of formal power series in $h \in \mathbf{C}$ with coefficients belonging to finite-dimensional subspaces of B . Both $B[[h]]$ and tensor products like $B[[h]] \otimes B[[h]]$ will be understood to be completed in the h -adic topology. The symbol $U_h\mathbf{g}$ [10] will denote the algebra on

^dFor multidimensional Heisenberg algebras the non-trivial task there was to find an intermediate transformation to a set of mutually commuting *pairs* $\{\alpha^i, \alpha_i^+\}$ of deformed generators; these generators α^i, α_i^+ are covariant neither under the $U\mathbf{g}$ nor the $U_h\mathbf{g}$ action.

the ring $\mathbf{C}[[h]]$ underlying the quantum group (completed in the h -adic topology).

2.1 Twisting groups into quantum groups

Let $H = (U\mathfrak{g}, m, \Delta, \varepsilon, S)$ be the cocommutative Hopf algebra associated to the universal enveloping (UE) algebra $U\mathfrak{g}$ of a Lie algebra \mathfrak{g} . The symbol m denotes the multiplication (in the sequel it will be dropped in the obvious way $m(a \otimes b) \equiv ab$, unless explicitly required), whereas Δ, ε, S the comultiplication, counit and antipode respectively. Assume that $H_h = (U_h\mathfrak{g}, m_h, \Delta_h, \varepsilon_h, S_h, \mathcal{R})$ is a quasitriangular non-cocommutative deformation of H [10]; $h \in \mathbf{C}$ denotes the deformation parameter, $m_h, \Delta_h, \varepsilon_h, S_h$ the deformed multiplication, comultiplication, counit and antipode respectively, and $\mathcal{R} \in U_h\mathfrak{g} \otimes U_h\mathfrak{g}$ the universal R -matrix.

A well-known theorem by Drinfel'd, Proposition 3.16 in Ref. [11] (whose results are to a certain extent already implicit in preceding works by Kohno [17]), proves, for any simple finite-dimensional Lie algebra \mathfrak{g} , the following results. There exists:

1. an algebra isomorphism $\varphi_h : U_h\mathfrak{g} \rightarrow U\mathfrak{g}[[h]]$ and
2. a 'twist', *i.e.* an element $\mathcal{F} \in U\mathfrak{g}[[h]] \otimes U\mathfrak{g}[[h]]$ satisfying the relations

$$(\varepsilon \otimes \text{id})\mathcal{F} = \mathbf{1} = (\text{id} \otimes \varepsilon)\mathcal{F} \quad (2.1.1)$$

$$\mathcal{F} = \mathbf{1} \otimes \mathbf{1} + O(h) \quad (2.1.2)$$

($\mathbf{1}$ denotes the unit of $U\mathfrak{g}$; (2.1.2) implies that \mathcal{F} is invertible as a formal power series in h)

such that H_h can be obtained from H through the following equations. Let $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$, $\mathcal{F}^{-1} = \mathcal{F}^{-1(1)} \otimes \mathcal{F}^{-1(2)}$, in a Sweedler's notation with *upper* indices; in the RHS a sum of many terms is implicitly understood, *e.g.* $\sum_i \mathcal{F}_i^{(1)} \otimes \mathcal{F}_i^{(2)}$. Then

$$m_h = \varphi_h^{-1} \circ m \circ (\varphi_h \otimes \varphi_h), \quad (2.1.3)$$

$$\varepsilon_h := \varepsilon \circ \varphi_h, \quad (2.1.4)$$

$$\Delta_h(x) = (\varphi_h^{-1} \otimes \varphi_h^{-1})[\mathcal{F}\Delta[\varphi_h(x)]\mathcal{F}^{-1}], \quad (2.1.5)$$

$$S_h(x) = \varphi_h^{-1}[\gamma^{-1}S[\varphi_h(x)]\gamma], \quad (2.1.6)$$

$$\mathcal{R} = [\varphi_h^{-1} \otimes \varphi_h^{-1}](\mathcal{F}_{21}q^{\frac{t}{2}}\mathcal{F}^{-1}). \quad (2.1.7)$$

Here $t := \Delta(\mathcal{C}) - \mathbf{1} \otimes \mathcal{C} - \mathcal{C} \otimes \mathbf{1}$ is the canonical invariant element in $U\mathfrak{g} \otimes U\mathfrak{g}$ (\mathcal{C} is the quadratic Casimir), the maps $m, \varepsilon, \Delta, S$ have been linearly extended from $U\mathfrak{g}$ to $U\mathfrak{g}[[h]]$, and

$$\gamma := S\mathcal{F}^{-1(1)} \cdot \mathcal{F}^{-1(2)}, \quad \gamma^{-1} = \mathcal{F}^{(1)} \cdot S\mathcal{F}^{(2)}. \quad (2.1.8)$$

Equation (2.1.3) says that $U\mathbf{g}[[h]]$ and $U_h\mathbf{g}$ are isomorphic through φ_h as algebras over $\mathbf{C}[[h]]$. Equation (2.1.4) says that, up to this isomorphism, ε_h and ε coincide. Equations (2.1.5), (2.1.6) say that, up to the same isomorphism, Δ_h, S_h differ from Δ, S by ‘similarity transformations’. Equation (2.1.5) is not in contradiction with the coassociativity of Δ and Δ_h^e , because the (nontrivial) coassociator

$$\phi := [(\Delta \otimes \text{id})(\mathcal{F}^{-1})](\mathcal{F}^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes \mathcal{F})[(\text{id} \otimes \Delta)(\mathcal{F})] \quad (2.1.9)$$

commutes with $\Delta^{(2)}(U\mathbf{g})$ (we denote by $\Delta^{(2)}$ the two-fold coproduct),

$$[\phi, \Delta^{(2)}(U\mathbf{g})] = 0. \quad (2.1.10)$$

From the properties of ϕ it follows also that $\gamma^{-1}\gamma' \in \text{Centre}(U\mathbf{g})$, and $S\gamma = \gamma'^{-1}$.

The above formulae can be read also in the other direction as giving a construction procedure of quasitriangular Hopf algebras. They can be also applied to *triangular* deformations of H [one needs only to set $t \equiv 0$ in (2.1.7)], that are quasitriangular deformations with a trivial coassociator, $\phi = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$. In fact the theorem cited above generalizes an older theorem [18], also by Drinfel’d.

The twist \mathcal{F} is defined (and unique) up to the transformation

$$\mathcal{F} \rightarrow \mathcal{F}T, \quad (2.1.11)$$

where T is a \mathbf{g} -invariant [*i.e.* commuting with $\Delta(U\mathbf{g})$] element of $U\mathbf{g}[[h]]^{\otimes 2}$ such that

$$T = \mathbf{1} \otimes \mathbf{1} + O(h), \quad (\varepsilon \otimes \text{id})T = \mathbf{1} = (\text{id} \otimes \varepsilon)T. \quad (2.1.12)$$

Under this transformation

$$\phi \rightarrow [(\Delta \otimes \text{id})(T^{-1})](T^{-1} \otimes \mathbf{1})\phi(\mathbf{1} \otimes T)[(\text{id} \otimes \Delta)(T)]. \quad (2.1.13)$$

A function

$$T = T(\mathbf{1} \otimes \mathcal{C}_i, \mathcal{C}_i \otimes \mathbf{1}, \Delta(\mathcal{C}_i)) \quad (2.1.14)$$

^eTo arrive at the above results Drinfel’d introduces the notion of quasitriangular quasi-Hopf algebra; the latter essentially involves the weakening of coassociativity of the coproduct into a property (“quasi-coassociativity”) valid only up to a similarity transformation through an element $\phi \in U\mathbf{g}[[h]]^{\otimes 3}$ (the “coassociator”). This notion is useful because quasitriangular quasi-Hopf algebras are mapped into each other under twists (even if the latter is not trivial). As an intermediate result, he shows that $U\mathbf{g}[[h]]$, beside the trivial quasitriangular quasi-Hopf structure $(U\mathbf{g}[[h]], m, \Delta, \varepsilon, S, \mathcal{R} \equiv \mathbf{1}^{\otimes 2}, \phi \equiv \mathbf{1}^{\otimes 3})$, has a non trivial one $(U\mathbf{g}[[h]], m, \Delta, \varepsilon, S, \mathcal{R} = q^{\frac{1}{2}}, \phi \neq \mathbf{1}^{\otimes 3})$.

of the Casimirs $\mathcal{C}_i \in U\mathfrak{g}$ of $U\mathfrak{g}$ and of their coproducts clearly is \mathfrak{g} -invariant. We find it plausible that any \mathfrak{g} -invariant T must be of this form; although we have found in the literature yet no proof of this conjecture, in the sequel we assume that this is true.

We will often use a ‘tensor notation’ for our formulae: $\mathcal{F}_{12,3} = (\Delta \otimes \text{id})\mathcal{F}_{12}$, $\mathcal{F}_{123,4} = (\Delta^{(2)} \otimes \text{id})\mathcal{F}_{12}$, and so on, and definition (2.1.9) reads $\phi \equiv \phi_{123} = \mathcal{F}_{12,3}^{-1}\mathcal{F}_{12}^{-1}\mathcal{F}_{23}\mathcal{F}_{1,23}$. ϕ satisfies the equations

$$\begin{aligned} q^{\frac{t_{13}+t_{23}}{2}} &= \phi_{231}^{-1} q^{\frac{t_{13}}{2}} \phi_{132} q^{\frac{t_{23}}{2}} \phi_{123}^{-1} \\ q^{\frac{t_{12}+t_{13}}{2}} &= \phi_{312} q^{\frac{t_{13}}{2}} \phi_{213}^{-1} q^{\frac{t_{12}}{2}} \phi_{123} \end{aligned} \quad (2.1.15)$$

(they follow from $(\Delta_h \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$, $(\text{id} \otimes \Delta_h)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$).

While for the twist \mathcal{F} , apart from its existence, very little explicit knowledge is available, Kohno [17] and Drinfel’d [11] have proved that, up to the transformation (2.1.13), ϕ is given by

$$\phi_m = \hat{g}^{-1}(x)\check{g}(x), \quad 0 < x < 1, \quad (2.1.16)$$

where $\hat{g}, \check{g}(x)$ are $U\mathfrak{g}[[h]]^{\otimes 3}$ -valued ‘analytic’^f solutions of the first order linear differential equation

$$\frac{dg}{dx} = \hbar \left(\frac{t_{12}}{x} + \frac{t_{23}}{x-1} \right) g, \quad 0 < x < 1 \quad (2.1.17)$$

($\hbar = \frac{h}{2\pi i}$) with the following asymptotic behaviour near the poles:

$$\hat{g} \cdot x^{-\hbar t_{12}} \xrightarrow{x \rightarrow 0} \mathbf{1}^{\otimes 3} \quad \check{g} \cdot (1-x)^{-\hbar t_{23}} \xrightarrow{x \rightarrow 1} \mathbf{1}^{\otimes 3}. \quad (2.1.18)$$

Using eq. (2.1.17) it is straightforward to verify that the RHS of eq. (2.1.16) is indeed independent of x .^g Using eq. (2.1.18) we can take the limit of eq. (2.1.16):

$$\phi_m = \lim_{x_0 \rightarrow 0^+} x_0^{-\hbar t_{12}} \check{g}(x_0). \quad (2.1.19)$$

We can formally solve equations (2.1.17), (2.1.18) for \check{g} by a path ordered integral,

$$\check{g}(x_0) = \lim_{y_0 \rightarrow 0^+} \left\{ \vec{P} \exp \left[-\hbar \int_{x_0}^{1-y_0} dx \left(\frac{t_{12}}{x} + \frac{t_{23}}{x-1} \right) \right] y_0^{\hbar t_{23}} \right\} \quad (2.1.20)$$

^fIn the sense that the coefficients $g_n(x)$ appearing in the expansion $g(x) = \sum_{n=0}^{\infty} g_n(x)h^n$ of g in h -powers are analytic functions of x with values in a finite-dimensional subspace of $U\mathfrak{g}^{\otimes 3}$.

^gKohno and Drinfel’d proved that ϕ can be obtained as the ‘monodromy’ of a system of three first order linear partial differential equations in three complex variables z_i (the so-called universal Knizhnik-Zamolodchikov [19] equations), with an $U\mathfrak{g}^{\otimes 3}$ -valued unknown f , $\frac{\partial f}{\partial z_i} = \hbar \sum_{j \neq i} \frac{t_{ij}}{z_i - z_j} f$. The system can be reduced to the equation (2.1.17) exploiting its invariance under linear transformations $z_i \rightarrow az_i + b$. For a review of these results see for instance Ref. [20].

$(\vec{P}[A(x)B(y)] := A(x)B(y)\vartheta(y-x) + B(y)A(x)\vartheta(x-y))$, whence

$$\phi_m = \lim_{x_0, y_0 \rightarrow 0^+} \left\{ x_0^{-\hbar t_{12}} \vec{P} \exp \left[-\hbar \int_{x_0}^{1-y_0} dx \left(\frac{t_{12}}{x} + \frac{t_{23}}{x-1} \right) \right] y_0^{\hbar t_{23}} \right\}. \quad (2.1.21)$$

Note that $\phi_m = \mathbf{1}^{\otimes 3} + O(\hbar^2)$. We will say that the twist \mathcal{F} is ‘*minimal*’ if the corresponding ϕ (2.1.9) is equal to ϕ_m or is trivial, respectively in the case of $H_h = U_h \mathfrak{g}$ or H_h is a triangular deformation of $U \mathfrak{g}$.

The algebra isomorphism $\varphi_h : U_h \mathfrak{g} \rightarrow U \mathfrak{g}[[\hbar]]$ is defined up to an inner automorphism (a ‘similarity transformation’) of $U \mathfrak{g}[[\hbar]]$,

$$\varphi_{h,v}(x) := v \varphi_h(x) v^{-1}, \quad (2.1.22)$$

with $v = \mathbf{1} + O(\hbar) \in U \mathfrak{g}[[\hbar]]$. It is easy to check that Drinfel’d theorem [11] remains true provided one replaces \mathcal{F} by $\mathcal{F}_v := (v \otimes v) \mathcal{F} \Delta(v^{-1})$ and all the objects derived from \mathcal{F} correspondingly; in particular, it is easy to check that the coassociator ϕ remains unchanged, because it is \mathfrak{g} -invariant

$$\phi_v = \Delta^{(2)}(v) \phi \Delta^{(2)}(v^{-1}) = \phi. \quad (2.1.23)$$

The freedom in choosing φ_h (and \mathcal{F}) is usually eliminated or reduced if one requires it to satisfy additional properties, such as to lead to a specific $*$ -structure for $U_h \mathfrak{g}$. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$ is the only \mathfrak{g} for which explicit φ_h ’s are known.

In the sequel we shall often use Sweedler’s notations with *lower* indices for the coproducts: $\Delta(x) \equiv x_{(1)} \otimes x_{(2)}$ for the cocommutative coproduct (in the RHS a sum $\sum_i x_{(1)}^i \otimes x_{(2)}^i$ of many terms is implicitly understood), $\Delta^{(n-1)}(x) \equiv x_{(1)} \otimes \dots \otimes x_{(n)}$ for the $(n-1)$ -fold cocommutative coproduct and $\Delta_h(x) \equiv x_{(\bar{1})} \otimes x_{(\bar{2})}$ (with barred indices) for the non-cocommutative one.

2.2 Deforming group-covariant Heisenberg algebras

The generic undeformed Heisenberg algebra is generated by the unit $\mathbf{1}_{\mathcal{A}}$ and elements a_i^+ and a^j satisfying the (anti)commutation relations

$$\begin{aligned} a^i a^j &= \pm a^j a^i \\ a_i^+ a_j^+ &= \pm a_j^+ a_i^+ \\ a^i a_j^+ &= \delta_j^i \mathbf{1}_{\mathcal{A}} \pm a_j^+ a^i \end{aligned} \quad (2.2.1)$$

(the \pm sign refers to Weyl and Clifford algebras respectively). a_i^+, a^j transform under the action of $U \mathfrak{g}$ according to some law

$$x \triangleright a_i^+ = \rho(x)_i^j a_j^+, \quad x \triangleright a^i = \rho(Sx)_j^i a^j; \quad (2.2.2)$$

here $x \in U\mathfrak{g}$ and ρ denotes some matrix representation of \mathfrak{g} . We shall call the corresponding algebra $\mathcal{A}_{\pm, \mathfrak{g}, \rho}$. When $x \in \mathfrak{g}$ the antipode reduces to $Sx = -x$. Clearly a^i belong to a representation of $U\mathfrak{g}$ which is the contragradient $\rho^\vee = \rho^T \circ S$ (T is the transpose) of the one of a_i^+ , ρ . Because of the linearity of the transformation (2.2.2) we shall also say that a_i^+ , a^i are “covariant”, or “tensors”, under \triangleright . The action \triangleright is extended to products of the generators using the standard rules of tensor product representations (technically speaking, using the coproduct Δ of the universal enveloping algebra $U\mathfrak{g}$, see formula (2.2.8) below), and then linearly to all of $\mathcal{A}_{\pm, \mathfrak{g}, \rho}$, $\triangleright : U\mathfrak{g} \times \mathcal{A}_{\pm, \mathfrak{g}, \rho} \rightarrow \mathcal{A}_{\pm, \mathfrak{g}, \rho}$; this is possible because the action of \mathfrak{g} is manifestly compatible with the commutation relations (2.2.1), and makes $\mathcal{A}_{\pm, \mathfrak{g}, \rho}$ into a (left) module algebra of (H, \triangleright) . In the sequel we shall denote by \triangleright also its linear extension to the corresponding algebras of power series in h .

For suitable ρ (specified below) $\mathcal{A}_{\pm, \mathfrak{g}, \rho}$ admits a deformation $\mathcal{A}_{\pm, \mathfrak{g}, \rho}^h$ with the same Poincaré series and the following features. $\mathcal{A}_{\pm, \mathfrak{g}, \rho}^h$ is generated by the unit $\mathbf{1}_{\mathcal{A}_h}$ and elements \tilde{A}_i^+ , \tilde{A}^i fulfilling deformed commutation relations (DCR) which can be put in the form

$$\tilde{A}^i \tilde{A}^j = \pm P_{hk}^{Fji} \tilde{A}^k \tilde{A}^h \quad (2.2.3)$$

$$\tilde{A}_i^+ \tilde{A}_j^+ = \pm P_{ij}^{Fhk} \tilde{A}_h^+ \tilde{A}_k^+ \quad (2.2.4)$$

$$\tilde{A}^i \tilde{A}_j^+ = \delta_j^i \mathbf{1}_{\mathcal{A}} \pm \tilde{P}^F i h_{jk} \tilde{A}_h^+ \tilde{A}^k \quad (2.2.5)$$

(as before, the upper and lower sign refer to Weyl and Clifford algebras respectively) and transforms under the action $\tilde{\triangleright}_h$ of $U_h\mathfrak{g}$ according to the law

$$x \tilde{\triangleright}_h \tilde{A}_i^+ = \rho_h^j(x) \tilde{A}_j^+, \quad x \tilde{\triangleright}_h \tilde{A}^i = \rho_h^i(S_h x) \tilde{A}^i. \quad (2.2.6)$$

Here $x \in U_h\mathfrak{g}$, ρ_h is the quantum group deformation of ρ , whereas P^F and \tilde{P}^F are two suitable quantum-group-covariant deformations of the ordinary permutator matrix P (by definition $P_{hk}^{ij} = \delta_k^i \delta_h^j$). By a redefinition (2.1.22) one can always choose a φ_h such that

$$\rho_h = \rho \circ \varphi_h. \quad (2.2.7)$$

\tilde{A}^i belong to a representation of $U_h\mathfrak{g}$ which is the quantum group contragradient $\rho_h^\vee = \rho_h^T \circ S_h$ of the one of \tilde{A}_i^+ , ρ_h . Because of the linearity of the transformation (2.2.6) we shall also say that \tilde{A}_i^+ , \tilde{A}^i are “covariant”, or “tensors”, under $\tilde{\triangleright}_h$. The action $\tilde{\triangleright}_h$ is extended to products of the generators by the formula

$$x \tilde{\triangleright}_h (ab) = (x_{(1)} \tilde{\triangleright}_h a) (x_{(2)} \tilde{\triangleright}_h b) \quad (2.2.8)$$

and then linearly to all of $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$, $\tilde{\triangleright}_h : U_h \mathbf{g} \times \mathcal{A}_{\pm, \mathbf{g}, \rho}^h \rightarrow \mathcal{A}_{\pm, \mathbf{g}, \rho}^h$; this is possible because the action $\tilde{\triangleright}_h$ is compatible with the above commutation relations, and makes $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$ into a (left) module algebra of $(H_h, \tilde{\triangleright}_h)$. The latter means also that

$$(xy) \tilde{\triangleright}_h a = x \tilde{\triangleright}_h (y \tilde{\triangleright}_h a) \quad (2.2.9)$$

$\forall x, y \in U_h \mathbf{g}$. In the undeformed setting the formulae corresponding to the previous two are obtained by just replacing $\tilde{\triangleright}_h$ by \triangleright and $\Delta_h(x) \equiv x_{(\bar{1})} \otimes x_{(\bar{2})}$ by the cocommutative coproduct $\Delta(x) \equiv x_{(1)} \otimes x_{(2)}$.

Up to our knowledge, only the following deformed algebras with the same Poincaré series as their classical counterpart have been constructed: $\mathcal{A}_{+, sl(N), \rho_d}^h$, [6, 7] $\mathcal{A}_{-, sl(N), \rho_d}^h$, [21] $\mathcal{A}_{+, so(N), \rho_d}^h$, [8], $\mathcal{A}_{-, sp(\frac{N}{2}), \rho_d}^h$ [22] (for each of the above Lie algebras ρ_d denotes the defining, N -dim representation) as well as those $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$ where ρ is the direct sum of many copies (m , say) of ρ_d , $\rho = \bigoplus_{\mu=1}^m \rho_{d, \mu}$ [22, 23]^h.

Explicitly, in the case $\rho = \rho_d$ the matrix \tilde{P}^F is equal to $q\hat{R}$ (or to its inverse), where \hat{R} is the braid matrix [24] of $U_h \mathbf{g}$,

$$\hat{R} := c_{\mathbf{g}} P \left[(\rho_{d, h})^{\otimes 2}(\mathcal{R}) \right] \quad c_{\mathbf{g}} := \begin{cases} q^{\frac{1}{N}} & \text{if } \mathbf{g} = sl(N) \\ 1 & \text{otherwise} \end{cases} \quad (2.2.10)$$

(here $\rho_{d, h}$ denotes the deformation of ρ_d and we have introduced the factor $c_{\mathbf{g}}$ to match to the conventional normalization), whereas P^F is a suitable first or second degree polynomial in \hat{R} . From formula (2.1.7) it then follows that

$$P^F = F U F^{-1} \quad (2.2.11)$$

$$\tilde{P}^F = F V F^{-1} \quad (2.2.12)$$

where $F := \rho_d^{\otimes 2}(\mathcal{F})$, $V = c_{\mathbf{g}} P q^{\rho_d^{\otimes 2}(t/2)}$ and U is also a polynomial in $P q^{\rho_d^{\otimes 2}(t/2)}$. One may actually choose $U = P$ without affecting the commutation relations, since in this case one can easily show that $(\mathbf{1} \mp U) \propto (\mathbf{1} \mp P)$, the (anti)symmetric projector. For later use we recall that from the projector decomposition and the properties of \hat{R} [24, 7, 8] it follows that the ‘ q -number operator’ $\tilde{\mathcal{N}} := \tilde{A}_i^+ \tilde{A}^i$ of $\mathcal{A}_{\pm, sl(N), \rho_d}^h$ satisfies the relations

$$\tilde{\mathcal{N}} \tilde{A}_i^+ = \tilde{A}_i^+ + q^{\pm 2} \tilde{A}_i^+ \tilde{\mathcal{N}} \quad \tilde{\mathcal{N}} \tilde{A}^i = q^{\mp 2} (-\tilde{A}^i + \tilde{A}^i \tilde{\mathcal{N}}), \quad (2.2.13)$$

^hFor a generic ρ , second degree relations of the type (2.2.3-2.2.5) may be not enough for a consistent definition of $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$; more precisely associativity may require the introduction of third or higher degree relations, which have no classical counterpart. In this case the Poincaré series of $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$ will be smaller than its classical counterpart, and $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$ will be physically not so interesting.

and the invariant elements $\tilde{A}^+ C \tilde{A}^+ := \tilde{A}_i^+ C^{ij} \tilde{A}_j^+$, $\tilde{A} C \tilde{A} := \tilde{A}^i C_{ji} \tilde{A}^j$ of $\mathcal{A}_{+,so(N)\rho_d}^h$ satisfy the relations

$$(\tilde{A} C \tilde{A}) \tilde{A}^i - \tilde{A}^i (\tilde{A} C \tilde{A}) = 0 \quad (2.2.14)$$

$$(\tilde{A}^+ C \tilde{A}^+) \tilde{A}_i^+ - \tilde{A}_i^+ (\tilde{A}^+ C \tilde{A}^+) = 0 \quad (2.2.15)$$

$$(\tilde{A} C \tilde{A}) \tilde{A}_i^+ - q^2 \tilde{A}_i^+ (\tilde{A} C \tilde{A}) = (1+q^{2-N}) C_{ij} \tilde{A}^j \quad (2.2.16)$$

$$\tilde{A}^i (\tilde{A}^+ C \tilde{A}^+) - q^2 (\tilde{A}^+ C \tilde{A}^+) \tilde{A}^i = (1+q^{2-N}) C^{ij} \tilde{A}_j^+. \quad (2.2.17)$$

In the case that ρ is the direct sum of many copies (say m) of ρ_d the commutation relations between the different copies are not trivial. We refer the reader to Ref. [22] for the explicit form of P^F, \tilde{P}^F . The important point here is that one can show that also in this case these matrices can be put in the form (2.2.11), (2.2.12), where

$$F := \rho^{\otimes 2}(\mathcal{F}), \quad (2.2.18)$$

and U, V are suitable $mN \times mN$ matrices such that

$$[U, \rho^{\otimes 2}(\Delta(U\mathbf{g}))] = [V, \rho^{\otimes 2}(\Delta(U\mathbf{g}))] = 0. \quad (2.2.19)$$

3 Realization of the quantum group action and of candidates for the deformed generators

It is immediate to check that one can define a Lie algebra homomorphism $\sigma : \mathbf{g} \rightarrow \mathcal{A}_{\pm, \mathbf{g}, \rho}$ by setting

$$\sigma(x) := \rho(x)_j^i a_i^+ a^j \quad (3.1)$$

for all $X \in \mathbf{g}$, and therefore extend it to all of $U\mathbf{g}$ as an algebra homomorphism $\sigma : U\mathbf{g} \rightarrow \mathcal{A}_{\pm, \mathbf{g}, \rho}$ by setting on the unit element $\sigma(\mathbf{1}_{U\mathbf{g}}) := \mathbf{1}_{\mathcal{A}}$. σ can be seen as the generalization of the Jordan-Schwinger realization of $\mathbf{g} = su(2)$ [25]

$$\sigma(j_+) = a_{\uparrow}^+ a^{\downarrow}, \quad \sigma(j_-) = a_{\downarrow}^+ a^{\uparrow}, \quad \sigma(j_0) = \frac{1}{2}(a_{\uparrow}^+ a^{\uparrow} - a_{\downarrow}^+ a^{\downarrow}). \quad (3.2)$$

Extending σ linearly to the corresponding algebras of power series in h , we can define also an algebra homomorphism

$$\sigma_{\varphi_h} := \sigma \circ \varphi_h : U_h \mathbf{g} \rightarrow \mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]]. \quad (3.3)$$

Since we know that a deforming map exists, although we cannot write it explicitly we can say that it must be possible to construct the map \triangleright_h defined in the

introduction. Our first step is to guess such a realization of $\tilde{\triangleright}_h$ on $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$. This requires fulfilling the conditions (2.2.8), (2.2.9), which characterize a left module algebra. There is a simple way to find such a realization, namely

Proposition 1 [13] *The (left) action $\triangleright_h : U_h \mathbf{g} \times \mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]] \rightarrow \mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]]$ can be realized in an ‘adjoint-like’ way:*

$$x \triangleright_h a := \sigma_{\varphi_h}(x_{(\bar{1})}) a \sigma_{\varphi_h}(S_h x_{(\bar{2})}). \quad (3.4)$$

Using the basic axioms characterizing the coproduct, counit, antipode in a generic Hopf algebra it is easy to check that (2.2.8), (2.2.9) are indeed fulfilled. The realization (3.4) is suggested by the cocommutative case, where it reduces to

$$x \triangleright a = \sigma(x_{(1)}) a \sigma(Sx_{(2)}). \quad (3.5)$$

Our second step is to realize elements $A_i^i, A_j^+ \in \mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]]$ that transform under (3.4) as $\tilde{A}^i, \tilde{A}_j^+$ in (2.2.6). Note that a^i, a_j^+ do *not* transform in this way. We recall

Proposition 2 [13] *Let \mathcal{F} be a twist associated to φ_h . For any choice of \mathbf{g} -invariant elements $u, \hat{u}, v, \hat{v} = \mathbf{1}_{\mathcal{A}} + O(h)$ in $\mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]]$ (in particular if they are trivial) the elements*

$$\begin{aligned} A_i^+ &:= \sigma(\mathcal{F}^{(1)}) v a_i^+ \hat{v} \sigma(S\mathcal{F}^{(2)} \gamma) \\ A^i &:= \sigma(\gamma' S\mathcal{F}^{-1(2)}) u a^i \hat{u} \sigma(\mathcal{F}^{-1(1)}) \end{aligned} \quad (3.6)$$

transform under (3.4) as $\tilde{A}^i, \tilde{A}_j^+$ in (2.2.6), and go to a_i^+, a^i in the limit $h \rightarrow 0$.ⁱ

Looking for \mathbf{g} -invariants making A^i, A_i^+ fulfil the DCR will be the third step of the construction (Sect. 5).

Remark 1. Without loss of generality, we can assume \mathcal{F} in definitions (3.6) to be minimal. In fact, since any other twist can be written in the form $\mathcal{F}T$ with $T \equiv T^{(1)} \otimes T^{(2)}$ as in (2.1.14), one finds that

$$\sigma(T^{(1)})(v a_i^+ \hat{v}) \sigma(ST^{(2)}) = v' a_i^+ \hat{v}' \quad (3.7)$$

$$\sigma(ST^{-1(1)})(u a^i \hat{u}) \sigma(T^{-1(2)}) = u' a^i \hat{u}' \quad (3.8)$$

with $u', \hat{u}', v', \hat{v}' = \mathbf{1} + O(h)$ also \mathbf{g} -invariant. This follows from the following observations: first, $\sigma(\mathcal{C}_i), \sigma(S\mathcal{C}_i)$ are \mathbf{g} -invariant and central in $\sigma(U\mathbf{g}[[h]])$, therefore the dependence of T on $\mathcal{C}_i \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{C}_i$ translates just into some replacements

ⁱ The Ansatz (3.6) has some resemblance with the one in Ref. [26], prop. 3.3, which defines an intertwiner $\alpha : U\mathbf{g}[[h]] \rightarrow U_h \mathbf{g}$ of $U_h \mathbf{g}$ -modules.

$u \rightarrow u', v \rightarrow v'$, etc; second, $(v a_i^+ \hat{v})$ transforms under \triangleright exactly as a_i^+ , whence $\sigma(\mathcal{C}_{i(1)})(v a_i^+ \hat{v})\sigma(S\mathcal{C}_{i(2)}) \stackrel{(3.5)}{=} \mathcal{C}_i \triangleright (v a_i^+ \hat{v}) = c_i(v a_i^+ \hat{v})$, where $c_i \in \mathbf{C}$ is the value of the Casimir \mathcal{C}_i in (the irreducible component of) the representation ρ to which a_i^+ belongs, and similarly for $(u a^i \hat{u})$, in other words the dependence of T on $\mathcal{C}_{i(1)} \otimes \mathcal{C}_{i(2)}$ translates just into multiplication of the generators by a constant.

Remark 2. Note that if ρ is reducible the previous proposition holds also if we allow for *different* invariants u, \hat{u}, v, \hat{v} within each irreducible component of ρ .

In the sequel we shall often use the compact notation

$$\mathbf{a}_i^+ := (v a_i^+ \hat{v}) \quad \mathbf{a}^i := (u a^i \hat{u}). \quad (3.9)$$

In the appendix we prove the following

Lemma 1 *If \mathcal{F} is a ‘minimal’, then*

$$\mathcal{F} = \gamma^{-1}(S\mathcal{F}^{-1(1)})\mathcal{F}_{(1)}^{-1(2)} \otimes \mathcal{F}_{(2)}^{-1(2)} \quad (3.10)$$

$$= \mathcal{F}_{(1)}^{-1(1)} \otimes \gamma'(S\mathcal{F}^{-1(2)})\mathcal{F}_{(1)}^{-1(2)} \quad (3.11)$$

$$\mathcal{F}^{-1} = \mathcal{F}_{(1)}^{(1)} \otimes \mathcal{F}_{(2)}^{(1)}(S\mathcal{F}^{(2)})\gamma \quad (3.12)$$

$$= \mathcal{F}_{(1)}^{(2)}(S\mathcal{F}^{(1)})\gamma'^{-1} \otimes \mathcal{F}_{(2)}^{(2)}. \quad (3.13)$$

We can find now useful alternative expressions for A_i^+, A^i .

Proposition 3 *With a ‘minimal’ \mathcal{F} , definitions (3.6) amount to*

$$A_i^+ = \mathbf{a}_l^+ \sigma(\mathcal{F}^{-1(2)})\rho(\mathcal{F}^{-1(1)})_i^l \quad (3.14)$$

$$A_i^+ = \rho(S\mathcal{F}^{(1)}\gamma'^{-1})_i^l \sigma(\mathcal{F}^{(2)})\mathbf{a}_l^+ \quad (3.15)$$

$$A^i = \rho(\mathcal{F}^{(1)})_i^l \sigma(\mathcal{F}^{(2)})\mathbf{a}^l \quad (3.16)$$

$$A^i = \mathbf{a}^l \sigma(\mathcal{F}^{-1(2)})\rho(\gamma^{-1}S\mathcal{F}^{-1(1)})_i^l. \quad (3.17)$$

Remark 3. In spite of its original definition (3.4), from the latter expressions we realize that only a ‘semiuniversal form’ of the type $(\rho \otimes \text{id})\mathcal{F}^{\pm 1}$ for \mathcal{F} is involved in the definition of A^i, A_j^+ .

Proof of Prop. 3. Observing that

$$\sigma(x)a = \sigma(x_{(1)})a\sigma(Sx_{(2)} \cdot x_{(3)}) \quad (3.18)$$

$$a\sigma(x) = \sigma(x_{(3)}Sx_{(2)})a\sigma(x_{(1)}) \quad (3.19)$$

for all $x \in U\mathfrak{g}$, $a \in \mathcal{A}_{\pm, \mathfrak{g}, \rho}$, we find

$$A_i^+ \stackrel{(3.5), (2.2.2)}{=} \mathbf{a}_l^+ \sigma \left[\mathcal{F}_{(2)}^{(1)}(S\mathcal{F}^{(2)})\gamma \right] \rho(\mathcal{F}_{(1)}^{(1)})_i^l \stackrel{(3.12)}{=} \mathbf{a}_l^+ \sigma(\mathcal{F}_1^{-1(2)})\rho(\mathcal{F}_1^{-1(1)})_i^l, \quad (3.20)$$

$$A^i \stackrel{(3.5), (2.2.2)}{=} \rho(\mathcal{F}_{(1)}^{-1(1)})_i^l \sigma \left[\gamma'(S\mathcal{F}^{-1(2)})\mathcal{F}_{(2)}^{-1(1)} \right] \mathbf{a}^l \stackrel{(3.11)}{=} \rho_i^l(\mathcal{F}_2^{(1)})\sigma(\mathcal{F}_2^{(2)})\mathbf{a}^l. \quad (3.21)$$

Similarly one proves the other relations. \square

4 Classical versus quantum group invariants

We have defined two actions $\triangleright, \triangleright_h$ on $\mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]]$. Their respective invariant subalgebras are respectively defined by

$$\mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}[[h]] := \{I \in \mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]] \mid x \triangleright I = \varepsilon(x)I \quad \forall x \in U\mathbf{g}\} \quad (4.1)$$

$$\mathcal{A}_{\pm, \mathbf{g}, \rho}^{h, inv}[[h]] := \{I \in \mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]] \mid x \triangleright_h I = \varepsilon_h(x)I \quad \forall x \in U_h\mathbf{g}\}. \quad (4.2)$$

If f is a deforming map corresponding to \triangleright_h , the second subalgebra clearly contains $f(\mathcal{A}_{\pm, \mathbf{g}, \rho}^{h, inv})$, where $\mathcal{A}_{\pm, \mathbf{g}, \rho}^{h, inv}$ denotes the $U_h\mathbf{g}$ -invariant subalgebra of $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$. What is the relation between $\mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}[[h]]$ and $\mathcal{A}_{\pm, \mathbf{g}, \rho}^{h, inv}[[h]]$?

Proposition 4 $\mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}[[h]] = \mathcal{A}_{\pm, \mathbf{g}, \rho}^{h, inv}[[h]]$.

Proof. We show that both subalgebras coincide with the one

$$\{I \in \mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]] \mid [\sigma(U\mathbf{g}), I] = 0\}. \quad (4.3)$$

Given any $I \in \mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}[[h]]$, $y \in U\mathbf{g}$, we find

$$\begin{aligned} I\sigma(y) &= I\varepsilon(y_{(1)})\sigma(y_{(2)}) \stackrel{(4.1)}{=} (y_{(1)} \triangleright I)\sigma(y_{(2)}) \\ &\stackrel{(3.5)}{=} \sigma(y_{(1)})I\sigma(Sy_{(2)} \cdot y_{(3)}) = \sigma(y_{(1)})I\varepsilon(y_{(2)}) = \sigma(y)I. \end{aligned}$$

this proves that (4.3) contains $\mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}[[h]]$. Conversely, if $[I, \sigma(U\mathbf{g})] = 0$ then

$$x \triangleright I \stackrel{(3.5)}{=} \sigma(x_{(1)})I\sigma(Sx_{(2)}) = I\sigma(x_{(1)} \cdot Sx_{(2)}) = I\varepsilon(x)$$

for any $x \in U\mathbf{g}$, proving that the set (4.3) is contained in $\mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}[[h]]$. Replacing in the previous arguments $\sigma, \varepsilon, \triangleright$ and $\Delta(x) \equiv x_{(1)} \otimes x_{(2)}$ by $\sigma_{\varphi_h}, \varepsilon_h, \triangleright_h$ and $\Delta_h(x) \equiv x_{(1)} \otimes x_{(2)}$, one proves that also $\mathcal{A}_{\pm, \mathbf{g}, \rho}^{h, inv}[[h]]$ coincides with the algebra (4.3). \square

In other words, the proposition states that invariants under the \mathbf{g} -action \triangleright are also $U_h\mathbf{g}$ -invariants (under \triangleright_h), and conversely, although in general \mathbf{g} -covariant objects (*i.e.* tensors) and $U_h\mathbf{g}$ -covariant ones do not coincide.

Since \triangleright (resp. \triangleright_h) acts in a linear and homogeneous way on the generators a^i, a_j^+ (resp. A^i, A_j^+), in the vector space $\mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}[[h]] = \mathcal{A}_{\pm, \mathbf{g}, \rho}^{h, inv}[[h]]$ we can choose a basis $\{I^n\}_{n \in \mathbf{N}}$ (resp. $\{I_h^n\}_{n \in \mathbf{N}}$) consisting of normal ordered homogeneous polynomials in a^i, a_j^+ (resp. A^i, A_j^+). For all \mathbf{g} the simplest invariant is the ‘number of particle operator’ $n \equiv I^1 := a_i^+ a^i$ and its deformed counterpart $\mathcal{N} \equiv I_h^1 = A_i^+ A^i$. In general the invariants will take the form

$$I^n := a_{j_1}^+ \dots a_{j_{k_n}}^+ d_{i_1 \dots i_{h_n}}^{j_1 \dots j_{k_n}} a^{i_1} \dots a^{i_{h_n}} \quad (4.4)$$

$$I_h^n := A_{j_1}^+ \dots A_{j_{k_n}}^+ D_{i_1 \dots i_{h_n}}^{j_1 \dots j_{k_n}} A^{i_1} \dots A^{i_{h_n}} \quad (4.5)$$

$(k_n, h_n \in \mathbf{N} \cup \{0\})$; the coefficients $d_{i_1 \dots i_{h_n}}^{j_1 \dots j_{k_n}}$ (resp. $D_{i_1 \dots i_{h_n}}^{j_1 \dots j_{k_n}}$) depend on the particular \mathbf{g} picked up and make up classical (resp. quantum) group isotropic tensors, *i.e.* satisfy

$$\left[\left(\rho^{\otimes k_n} \otimes \rho^{\vee \otimes h_n} \right) \left(\Delta^{(b_n-1)}(x) \right) \right]_{J'_n I'_n}^{J_n I_n} d_{I'_n}^{J'_n} = \varepsilon(x) d_{I_n}^{J_n} \quad (4.6)$$

$$\left[\left(\rho_h^{\otimes k_n} \otimes \rho_h^{\vee \otimes h_n} \right) \left(\Delta_h^{(b_n-1)}(y) \right) \right]_{J'_n I'_n}^{J_n I_n} D_{I'_n}^{J'_n} = \varepsilon_h(y) D_{I_n}^{J_n} \quad (4.7)$$

$\forall x \in U\mathbf{g}[[h]], y \in U_h\mathbf{g}$. Here and in the rest of the section we use the collective-index notation $I_n \equiv (i_1 \dots i_{h_n})$, $J_n \equiv (j_1 \dots j_{k_n})$ and the short-hand notation $b_n := h_n + k_n$. Using formula (2.1.6) it is straightforward to verify that the d 's and D 's are related to each other by

$$D_{I_n}^{J_n} \propto \left[\left(\rho^{\otimes k_n} \otimes \rho^{\vee \otimes h_n} \right) \left((\mathbf{1}^{\otimes k_n} \otimes (\gamma'^{-1})^{\otimes h_n}) \mathcal{F}_{12 \dots b_n} \right) \right]_{J'_n I'_n}^{J_n I_n} d_{I'_n}^{J'_n}, \quad (4.8)$$

where $\mathcal{F}_{12 \dots b} \in U\mathbf{g}[[h]]^{\otimes b}$ is an intertwiner between $\Delta^{(b-1)}$ and $\Delta_h^{(b-1)}$ and is given, up to multiplication from the right by a \mathbf{g} -invariant tensor $Q \in U\mathbf{g}[[h]]^{\otimes b_n}$, by

$$\mathcal{F}_{12 \dots b} = \mathcal{F}_{(b-1)b} \mathcal{F}_{b-2, (b-1)b} \dots \mathcal{F}_{1, 2 \dots b}. \quad (4.9)$$

The isotropic tensors corresponding to I^1, I_h^1 are $d_i^j = \delta_i^j = D_j^i$. The replacement $\mathcal{F} \rightarrow \mathcal{F} \cdot T$, with $T \in U\mathbf{g}[[h]]^{\otimes 2}$ and \mathbf{g} -invariant, results also in multiplication from the right by a related Q .

Relation (4.8) guarantees the existence of D 's in one-to-one correspondence with the d 's, but from the practical viewpoint is not of much help for finding the D 's (since the universal \mathcal{F} is unknown and its matrix representations are known only for few representations); the latter can be found more easily from the knowledge of \mathcal{R} and a direct study of $\tilde{\mathfrak{s}}_h$.

The question whether $\mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}$ is finitely generated, *i.e.* whether all I^n can be expressed as *polynomials* in a finite number of them, is part of an important problem originally raised by Hilbert^j. This is in general not the case. However, they can be expressed as *algebraic functions* of a finite number of them [27]. The classification of the latter (for arbitrary \mathbf{g}) is not completed yet [28].

We would like to ask here a different question. According to the above proposition, I_h^n can be expressed as a power series in h with coefficients in $\mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}$, in other words as a 'function' $I_h^n = m^n(h, \{I^n\})$. Since the latter are functions of a^i, a_j^+ , we can express I_h^n also as a function $I_h^n = f^n(h, a^i, a_j^+)$. How to find m^n, f^n ? We give an explicit answer to the second question by proving (see the appendix)

^jThe fourteenth problem proposed by Hilbert at the 1900 conference of mathematicians.

Proposition 5

$$I_h^n = (\mathbf{a}^+ \dots \mathbf{a}^+)_{M_n} (\mathbf{a} \dots \mathbf{a})^{L_n} \times \left[\left(\rho^{\otimes k_n} \otimes \rho^\vee \otimes^{\otimes h_n} \otimes \sigma \right) \left(\phi_{(b_n-1)b_n(b_n+1)}^{-1} \phi_{(b_n-2),(b_n-1)b_n,(b_n+1)}^{-1} \dots \phi_{1,2\dots b_n,(b_n+1)}^{-1} \right) \right]_{J_n I_n}^{M_n L_n} d_{I_n}^{J_n}$$

where $\phi_{1,2\dots m,m+1} := (id \otimes \Delta^{(m-2)} \otimes id) \phi_{123}$ and $b_n := h_n + k_n$.

Remark 4. Note that in these equations the whole dependence on the twist \mathcal{F} is concentrated in the coassociator ϕ of \mathbf{g} and in its coproducts. Consequently, use of formula (2.1.21) allows the explicit determination of the dependence of I_h^n 's on a^i, a_j^+ ; clearly, the latter will be in general highly non-polynomial.

Let us address now the first question. If H_h is *triangular* then ϕ^{-1} and all its coproducts are trivial, the \mathbf{g} -invariants u, v, \hat{u}, \hat{v} may be chosen trivial as well [13], and from the previous proposition we find $I_h^n = I^n$. If H_h is a *genuine quasitriangular* Hopf algebra, such as $U_h \mathbf{g}$, then in general $I_h^n \neq I^n$; the I_h^n will be some nontrivial function of the I^m 's, generally speaking highly non-polynomial, as well. This can be already verified for the simplest invariants. We will show in next section that *e.g.* $I_h^1 = (n)_{q^2} \equiv (I^1)_{q^2}$ in the $\mathbf{g} = sl(N)$ case. In general,

$$I_h^n = I^n + O(h). \quad (4.10)$$

This follows from $D_{J_n}^{I_n} = d_{J_n}^{I_n} + O(h)$, $A^i = a^i + O(h)$, $A_i^+ = a_i^+ + O(h)$.

In the $\mathbf{g} = so(N)$, $\rho = \rho_d$ case, beside δ_j^i , another basic isotropic tensor is the classical metric matrix $c_{ij} = c_{ji}$ (with inverse $c^{ij} = c^{ji}$, to which there corresponds the deformed metric matrix [24] C_{ij} , and its inverse C^{ij}):

$$C^{ij} = F_{hk}^{ij} c^{hk} = \rho_d(\gamma^{-1})_h^i c^{hj}. \quad C_{ij} = c_{ih} \rho_d(\gamma)_j^h; \quad (4.11)$$

the last two equalities follow from the $so(N)$ property

$$\rho_d(Sx)_j^i = \rho_d(x)_l^m c^{li} c_{mj} \quad x \in U\mathbf{g}. \quad (4.12)$$

So one can build the invariants

$$\begin{aligned} I^{2,0} &:= a^i c_{ij} a^j \equiv a c a & I^{0,2} &:= a_i^+ c^{ij} a_j^+ \equiv a^+ c a^+ \\ I_h^{2,0} &:= A^i C_{ji} A^j \equiv A C A & I_h^{0,2} &:= A_i^+ C^{ij} A_j^+ \equiv A^+ C A^+; \end{aligned} \quad (4.13)$$

we will see in next section that $I_h^{2,0} \neq I^{2,0}$, $I_h^{0,2} \neq I^{0,2}$. We leave the determination of the general dependence of I_h^n on I^m 's as a subject for further investigation.

5 Realization of the deformed generators

In section 3 we have left some freedom in the definition of A^i, A_i^+ : the \mathbf{g} -invariants u, \hat{u}, v, \hat{v} appearing in the definitions (3.9) of $\mathbf{a}^i, \mathbf{a}_i^+$ have not been specified. Can we choose them in such a way that A^i, A_i^+ fulfil the DCR (deformed commutation relations) of $\mathcal{A}_{\pm, \mathbf{g}, \rho}^h$? To explicitly study this question in the appendix we prove

Proposition 6 *If we replace $\tilde{A}^i, \tilde{A}_j^+ \rightarrow A^i, A_j^+$ [with A^i, A_j^+ defined as in formulae (3.14), (3.16), with a minimal \mathcal{F}], then equations (2.2.3), (2.2.4), (2.2.5) become equivalent to*

$$\mathbf{a}^i \mathbf{a}^j = \pm (M^{-1} U M)_{lm}^{ji} \mathbf{a}^m \mathbf{a}^l \quad (5.0.1)$$

$$\mathbf{a}_i^+ \mathbf{a}_j^+ = \pm \mathbf{a}_l^+ \mathbf{a}_m^+ (M^{-1} U M)_{ij}^{lm} \quad (5.0.2)$$

$$\mathbf{a}^i \mathbf{a}_j^+ = \delta_j^i \mathbf{1}_{\mathcal{A}} \pm \mathbf{a}_l^+ (M^{-1} V M)_{jm}^{il} \mathbf{a}^m \quad (5.0.3)$$

where $U \equiv \|U_{hk}^{ij}\|$, $V \equiv \|V_{hk}^{ij}\|$ are the (numerical) matrices introduced in equations (2.2.19) and $M \equiv \|M_{hk}^{ij}\|$ is the $\sigma(U\mathbf{g}[[h]])$ -valued matrix defined by

$$M := (\rho \otimes \rho \otimes \sigma)(\phi_m). \quad (5.0.4)$$

We recall that, if $\rho = \rho_d$, U is the permutation matrix P and $V \propto P q^{\rho_d^{\otimes 2}(\frac{1}{2})}$.

Remark 5. The above equations have to be understood as equations in the unknowns u, \hat{u}, v, \hat{v} . They can be studied explicitly because the whole dependence on \mathcal{F} is concentrated again in the coassociator ϕ of $U_h \mathbf{g}$.

Remark 6. If H_h is a triangular deformation, then $U = V = P$, $\phi = \mathbf{1}^{\otimes 3}$ (and consequently $M = \mathbf{1}^{\otimes 3}$), and the eq. (5.0.1), (5.0.2) are satisfied with trivial invariants u, \hat{u}, v, \hat{v} , i.e. with $\mathbf{a}^i = a^i$, $\mathbf{a}_i^+ = a_i^+$. This was already shown in Ref. [13].

To look for solutions of eq. (5.0.1), (5.0.2), (5.0.3) for genuine quasitriangular deformations we have to treat the \mathbf{g} 's belonging to different classical series separatly. We consider here $\mathcal{A}_{\pm, sl(N), \rho_d}^h, \mathcal{A}_{+, so(N), \rho_d}^h$.

5.1 The case of $\mathcal{A}_{\pm, sl(N), \rho_d}^h$

As a basis of \mathbf{g} we choose $\{E_{ij}\}_{i,j=1,\dots,N}$ with $\sum_{i=1}^N E_{ii} \equiv 0$ (so that there exist only $N^2 - 1$ linearly independent E_{ij}), satisfying

$$[E_{ij}, E_{hk}] = E_{ik} \delta_{jh} - E_{jh} \delta_{ik} \quad (5.1.1)$$

The quadratic Casimir reads

$$\mathcal{C} = E_{ij} E_{ji}, \quad (5.1.2)$$

implying

$$t = 2 E_{ij} \otimes E_{ji} \quad (5.1.3)$$

The matrix representation of E_{ij} in the fundamental representation ρ takes the form

$$\rho(E_{ij}) = e_{ij} - \frac{\delta_{ij}}{N} \mathbf{1}_N, \quad (5.1.4)$$

where e_{ij} is the $N \times N$ matrix with all vanishing entries but a 1 in the i -th row and j -th column, and $\mathbf{1}_N = \sum_i e_{ii}$ is the $N \times N$ unit matrix; whereas the Jordan-Schwinger realization takes the form

$$\sigma(E_{ij}) = a_i^+ a^j - \frac{\delta_{ij}}{N} n. \quad (5.1.5)$$

As a consequence $\sigma(\mathcal{C}) = n(N \pm n \mp 1) - \frac{n^2}{N}$. From the previous equations one finds

$$\begin{aligned} (\rho \otimes \rho \otimes \sigma) \left(\frac{t_{12}}{2} \right) &= e_{ij} \otimes e_{ji} \otimes \mathbf{1}_A - \frac{1}{N} \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_A =: P - \frac{\mathbf{1}^{\otimes 3}}{N}, \\ (\rho \otimes \rho \otimes \sigma) \left(\frac{t_{23}}{2} \right) &= \mathbf{1}_N \otimes e_{ij} \otimes a_j^+ a^i - \mathbf{1}_N \otimes \mathbf{1}_N \otimes \frac{n}{N} =: A - \mathbf{1}_N^{\otimes 2} \otimes \frac{n}{N} \\ (\rho \otimes \rho \otimes \sigma) \left(\frac{t_{13}}{2} \right) &= e_{ij} \otimes \mathbf{1}_N \otimes a_j^+ a^i - \mathbf{1}_N \otimes \mathbf{1}_N \otimes \frac{n}{N} =: B - \mathbf{1}_N^{\otimes 2} \otimes \frac{n}{N}; \end{aligned}$$

P denotes the permutation matrix on $\mathbf{C}^N \otimes \mathbf{C}^N$, multiplied by $\mathbf{1}_A$.

$n := a_i^+ a^i$ is an element of $\mathcal{A}_{\pm, sl(N), \rho_d}^{inv}[[h]]$. We try to solve eq. (5.0.1-5.0.3) with invariants u, \hat{u}, v, \hat{v} depending only on n . Using relations

$$[n, a_i^+] = a_i^+ \quad [n, a^i] = -a^i. \quad (5.1.6)$$

we can thus commute \hat{v} to the left of a^i and u to the right of a_i^+ in formula (3.9), and look for $\mathbf{a}^i, \mathbf{a}_i^+$ directly in the form $\mathbf{a}^i := I a^i, \mathbf{a}_i^+ := a_i^+ \tilde{I}$, with $I = I(n) \in \mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}[[h]]$, $\tilde{I} = \tilde{I}(n) \in \mathcal{A}_{\pm, \mathbf{g}, \rho}^{inv}[[h]]$. From eq. (3.14), (3.16) it follows $\mathcal{N} := A_i^+ A^i = \mathbf{a}_i^+ \mathbf{a}^i = n \hat{I}(n-1)$, where $\hat{I}(n) := I(n) \tilde{I}(n)$. In order that \mathcal{N}, A_i^+, A^i satisfies the commutations relations (2.2.13), we therefore require $\hat{I}(n) = \frac{(n+1)_{q^{\pm 2}}}{n+1}$, with $(x)_a := \frac{x^a - 1}{a - 1}$. Summing up, we leave $I(n)$ undetermined and we pick

$$\begin{aligned} \mathbf{a}^i &:= I a^i, & I &\equiv I(n), \\ \mathbf{a}_i^+ &:= a_i^+ \tilde{I} & \tilde{I}(n) &:= I^{-1}(n) \frac{(n+1)_{q^{\pm 2}}}{(n+1)}. \end{aligned} \quad (5.1.7)$$

These ansatz can also be written in the equivalent form

$$\mathbf{a}^i = u(n) a^i u^{-1}(n), \quad \mathbf{a}_i^+ = v(n) a_i^+ v^{-1}(n), \quad (5.1.8)$$

where u, v are constrained by the relation

$$u v^{-1} = y = y_{sl(N)} := \frac{\Gamma(n+1)}{\Gamma_{q^2}(n+1)} \quad (5.1.9)$$

and Γ, Γ_{q^2} are the Euler's Γ -function and its deformation (A.4.9).

We have now the right ansatz to show that the DCR of N -dimensional $U_h sl(N)$ -covariant Heisenberg algebra are fulfilled. In the appendix we prove

Theorem 1 *When $\mathfrak{g} = sl(N)$, the objects A^i, A_i^+ ($i = 1, 2, \dots, N$) defined in formulae (3.14), (3.16), (5.1.7) satisfy the corresponding DCR (2.2.3), (2.2.4), (2.2.5).*

In Ref. [13] the case $\mathfrak{g} = sl(2)$ was worked out explicitly. Choosing $u = v^{-1} = \sqrt{y_{sl(2)}}$, we found for the $A^i, A_i^+ \in \mathcal{A}_{+,sl(2),\rho_d}[[h]]$ ($i = \uparrow, \downarrow$),

$$\begin{aligned} A_{\uparrow}^+ &= \sqrt{\frac{(n^{\uparrow})_{q^2}}{n^{\uparrow}}} q^{n^{\downarrow}} a_{\uparrow}^+ & A_{\downarrow}^+ &= \sqrt{\frac{(n^{\downarrow})_{q^2}}{n^{\downarrow}}} a_{\downarrow}^+ \\ A^{\uparrow} &= a^{\uparrow} \sqrt{\frac{(n^{\uparrow})_{q^2}}{n^{\uparrow}}} q^{n^{\downarrow}} & A^{\downarrow} &= a^{\downarrow} \sqrt{\frac{(n^{\downarrow})_{q^2}}{n^{\downarrow}}}, \end{aligned} \quad (5.1.10)$$

and for the $A^i, A_i^+ \in \mathcal{A}_{-,sl(2),\rho_d}[[h]]$

$$\begin{aligned} A_{\uparrow}^+ &= q^{-n^{\downarrow}} a_{\uparrow}^+ & A_{\downarrow}^+ &= a_{\downarrow}^+ \\ A^{\uparrow} &= a^{\uparrow} q^{-n^{\downarrow}} & A^{\downarrow} &= a^{\downarrow}. \end{aligned} \quad (5.1.11)$$

Here $n_i := a_i^+ a^i$ (no sum over i). Let us compare the generators (5.1.10) with the ones found in Ref. [15]. In our notation the latter would read

$$\begin{aligned} A_{\alpha\uparrow}^+ &= q^{n^{\downarrow}} a_{\uparrow}^+ & A_{\alpha\downarrow}^+ &= a_{\downarrow}^+ \\ A^{\alpha\uparrow} &= a^{\uparrow} \frac{(n^{\uparrow})_{q^2}}{n^{\uparrow}} q^{n^{\downarrow}} & A^{\alpha\downarrow} &= a^{\downarrow} \frac{(n^{\downarrow})_{q^2}}{n^{\downarrow}}. \end{aligned} \quad (5.1.12)$$

It is straightforward to check that the element $\alpha \in \mathcal{A}_{\pm, \mathfrak{g}, \rho}[[h]]$ such that $A^{\alpha i} = \alpha A^i \alpha^{-1}$, $A_{\alpha i}^+ = \alpha A_i^+ \alpha^{-1}$ [formulae (1.1)] is

$$\alpha := \sqrt{\frac{\Gamma(n^{\uparrow} + 1) \Gamma(n^{\downarrow} + 1)}{\Gamma_{q^2}(n^{\uparrow} + 1) \Gamma_{q^2}(n^{\downarrow} + 1)}}, \quad (5.1.13)$$

where Γ is the Euler Γ -function and Γ_{q^2} its q -deformation (A.4.9).

5.2 The case of $\mathcal{A}_{+,so(N),\rho_d}^h$

As a basis of $\mathfrak{g} = so(N)$ we choose $\{L_{ij}\}_{i,j=1,\dots,N}$ with $L_{ij} = -L_{ji}$ (so that there exist only $\frac{N(N-1)}{2}$ linearly independent L_{ij}), satisfying

$$[L_{ij}, L_{hk}] = L_{ik} c_{jh} + L_{kj} c_{ih} - L_{hj} c_{ik} - L_{ih} c_{jk}; \quad (5.2.1)$$

here c_{ij} denotes the (classical) metric matrix on the N -dimensional Euclidean space ($c_{ij} = c_{ji}$), which in the special case we choose real Cartesian coordinates takes simply the form $c_{ij} = \delta_{ij}$. In the rest of this subsection classically-covariant indices will be lowered and raised by means of multiplication by c : $v_i = c_{ij}v^j$, $v^i = c^{ij}v_j$, etc., and $v \cdot w := v^i w^j c_{ij} = v_i w^i = v^i w_i$. The quadratic Casimir reads

$$\mathcal{C} = \frac{1}{2} L_{ij} L^{ji}, \quad (5.2.2)$$

implying

$$t = L_{ij} \otimes L^{ji} \quad (5.2.3)$$

The matrix representation of E_{ij} in the fundamental representation ρ takes the form

$$\rho(L_{ij}) = e_{ih} c_{hj} - e_{jh} c_{hi}, \quad (5.2.4)$$

and the Jordan-Schwinger realization becomes

$$l_{ij} := \sigma(L_{ij}) = a_i^+ a^h c_{hj} - a_j^+ a^h c_{hi} = a^h a_i^+ c_{hj} - a^h a_j^+ c_{hi}. \quad (5.2.5)$$

It is easy to work out

$$l^2 := \sigma(\mathcal{C}) + (1 - \frac{N}{2})^2 = \left(n + \frac{N}{2} - 1\right)^2 - (a^+ \cdot a^+)(a \cdot a), \quad (5.2.6)$$

and to check that, as expected

$$[l^2, a^+ \cdot a^+] = 0 = [l^2, a \cdot a] \quad (5.2.7)$$

A direct calculation also shows that

$$\begin{aligned} [l^2, a^i] &= -a^i(2n+1+N) + 2(a \cdot a) a_j^+ c^{ji} = -a^i(2n-3+N) + 2a_j^+ c^{ji}(a \cdot a) \\ [l^2, a_i^+] &= a_i^+(2n+3+N) - 2c_{ij} a^j (a^+ \cdot a^+) = a_i^+(2n-1+N) - 2c_{ij} (a^+ \cdot a^+) a^j. \end{aligned}$$

We look for “eigenvectors” of l^2

$$l^2 \alpha^i = \alpha^i \lambda \quad l^2 \alpha_i^+ = \alpha_i^+ \mu,$$

in the form $\alpha^i = a^i \gamma + a_j^+ c^{ji} (a \cdot a) \delta$, $\alpha_i^+ = a_i^+ \alpha + a^j c_{ji} (a^+ \cdot a^+) \beta$ with “eigenvalues” λ, μ and “coefficients” $\alpha, \beta, \gamma, \delta$ depending on n, l^2 . We find second order equations for λ, μ with solutions $\lambda, \mu = (l \pm 1)^2$, where formally $l = \sqrt{l^2}$. We can therefore consistently extend $\mathcal{A}_{+,so(N),\rho_d}$ by the introduction of a new generator l [whose square is constrained to give the l^2 defined in eq. (5.2.6)] such that

$$\alpha_\pm^i := a^i \left(n + \frac{N}{2} - 1 \pm l\right) - c^{ij} a_j^+ (a \cdot a) = a^i \left(n + \frac{N}{2} + 1 \pm l\right) - (a \cdot a) c^{ij} a_j^+,$$

$$\alpha_{i,\pm}^+ := a_i^+(n + \frac{N}{2} - 1 \pm l) - (a^+ \cdot a^+) c_{ij} a^j = a_i^+(n + \frac{N}{2} + 1 \pm l) - c_{ij} a^j (a^+ \cdot a^+)$$

satisfy

$$\begin{aligned} l \alpha_{i,\pm}^+ &= \alpha_{i,\pm}^+ (l \pm 1), \\ l \alpha_{i,\pm}^i &= \alpha_{i,\pm}^i (l \mp 1). \end{aligned} \quad (5.2.8)$$

After these preliminaries, let us determine the right $\mathbf{a}^i, \mathbf{a}_i^+$'s for A^i, A_i^+ to satisfy the DCR. To satisfy at once eq.'s (2.2.14), (2.2.15) we make the ansatz:

$$\mathbf{a}^i = u(n, l) a^i u^{-1}(n, l), \quad \mathbf{a}_i^+ = v(n, l) a_i^+ v^{-1}(n, l) \quad (5.2.9)$$

This implies

$$A^+ C A^+ \stackrel{(5.2.9)}{=} \mathbf{a}_l^+ \mathbf{a}_m^+ c^{lm} = v a^+ \cdot a^+ v^{-1} \quad (5.2.10)$$

$$A C A \stackrel{(5.2.9)}{=} \mathbf{a}^l \mathbf{a}^m c_{lm} = u a \cdot a u^{-1} \quad (5.2.11)$$

The DCR determine only the product $y := v^{-1}u$; we are going to show now that eq.'s (2.2.16), (2.2.17) completely determine the latter. It is immediate to check that the former implies

$$y \left[2c^{ij} a_j^+ (a^+ \cdot a^+) a^i \right] y^{-1}(n+2, l) - q^2 y(n-2, l) (a^+ \cdot a^+) a^i y^{-1} = (1+q^{N-2}) c^{ij} a_j^+.$$

Expressing $a_i^+, c_{ij} (a^+ \cdot a^+) a^j$ as combinations of $\alpha_{i\pm}$ we easily move y past the “eigenvectors” $\alpha_{i\pm}$ of n, l ; factoring out (from the right) $\frac{c_{ij}}{2l}$ we end up with a LHS being a combination of $\alpha_{i,+}, \alpha_{i,-}$. Therefore eq. (2.2.16) amounts to the condition that the corresponding coefficients vanish:

$$(1+q^{N-2}) = (n + \frac{N}{2} + 1 - l) y(n+1, l+1) y^{-1}(n+2, l) - q^2 (n + \frac{N}{2} - 1 - l) y(n-1, l-1) y^{-1}(n, l)$$

$$(1+q^{N-2}) = (n + \frac{N}{2} + 1 + l) y(n+1, l+1) y^{-1}(n+2, l) - q^2 (n + \frac{N}{2} - 1 - l) y(n-1, l-1) y^{-1}(n, l)$$

Similarly, from eq. (2.2.17) it follows

$$(1+q^{N-2}) = (n + \frac{N}{2} + 1 - l) y(n, l) y^{-1}(n+1, l-1) - q^2 (n + \frac{N}{2} - 1 - l) y(n-2, l) y^{-1}(n-1, l-1)$$

$$(1+q^{N-2}) = (n + \frac{N}{2} + 1 + l) y(n, l) y^{-1}(n+1, l+1) - q^2 (n + \frac{N}{2} - 1 + l) y(n-2, l) y^{-1}(n-1, l+1)$$

It is straightforward to check that the last four equations are solved by

$$u v^{-1} = y = y_{so(N)} := \left(\frac{1+q^{N-2}}{2} \right)^{-n} \frac{\Gamma \left[\frac{1}{2} \left(n + \frac{N}{2} + 1 - l \right) \right] \Gamma \left[\frac{1}{2} \left(n + \frac{N}{2} + 1 + l \right) \right]}{\Gamma_{q^2} \left[\frac{1}{2} \left(n + \frac{N}{2} + 1 - l \right) \right] \Gamma_{q^2} \left[\frac{1}{2} \left(n + \frac{N}{2} + 1 + l \right) \right]}, \quad (5.2.12)$$

where Γ, Γ_{q^2} are the Euler's Γ -function and its deformation (A.4.9).

We have now the right ansatz to fulfill the DCR of N -dimensional $U_h so(N)$ -covariant Weyl algebra. We state without proof the

Theorem 2 *When $\mathfrak{g} = so(N)$, the objects A^i, A_i^+ ($i = 1, 2, \dots, N$) defined in formulae (3.14), (3.16), (5.2.9), (5.2.12) satisfy the corresponding DCR (2.2.3), (2.2.4), (2.2.5).*

6 *-Structures

Given the Hopf $*$ -algebra $H_h = (U_h \mathfrak{g}, m, \Delta_h, \varepsilon_h, S_h, \mathcal{R}, *_h)$, we ask now whether the $*$ -structures \dagger_h of $\mathcal{A}_{\pm, \mathfrak{g}, \rho}^h$ compatible with the action \triangleright_h of $U_h \mathfrak{g}$, i.e. such that

$$(x \tilde{\triangleright}_h a)^{\dagger_h} = S_h^{-1}(x^{*h}) \tilde{\triangleright}_h a^{\dagger_h}, \quad (6.1)$$

can be naturally realized by the ones of $\mathcal{A}_{\pm, \mathfrak{g}, \rho}$.

We stick to the case that H_h is the compact real section of $U_h \mathfrak{g}$. Then $U_h \mathfrak{g}$ as an algebra is isomorphic to $U \hat{\mathfrak{g}}[[h]]$, where $\hat{\mathfrak{g}}$ is the compact section of \mathfrak{g} and $h \in \mathbf{R}$, and the trivializing maps φ_h intertwine between $*_h$ and $*$, $[\varphi_h(x)]^* = \varphi_h(x^{*h})$ where $*$ is the classical $*$ -structure in $U \mathfrak{g}$ having the elements of $\hat{\mathfrak{g}}$ as fixed points.

If \mathcal{F} is unitary then the corresponding γ, γ', ϕ clearly satisfy

$$\gamma' = \gamma^* \quad \phi^{*\otimes *\otimes *} = \phi^{-1}. \quad (6.2)$$

On the other hand, it is evident that the ‘minimal’ coassociator ϕ_m (2.1.16) is also unitary (because $h \in \mathbf{R}$); one could actually show that the unitary \mathcal{F} is also minimal.

If ρ_h is a $*$ -representation of H , the $*$ -structure $(\tilde{A}^i)^{\dagger_h} = \tilde{A}_i^+$ is clearly compatible with $\tilde{\triangleright}_h$ [condition (6.1)]; the classical counterpart of ρ_h is also a $*$ -representation ρ of H (i.e. $\rho(x^*) = \overline{\rho^T(x)}$), and formula

$$(a^i)^{\dagger} = a_i^+ \quad (6.3)$$

defines in $\mathcal{A}_{\pm, \mathfrak{g}, \rho}$ a $*$ -structure (‘hermitean conjugation’) \dagger compatible with \triangleright . Correspondingly, it is immediate to check that $\sigma, \sigma_{\varphi_h}$ become $*$, $*_h$ -homomorphisms respectively,

$$\sigma \circ * = \dagger \circ \sigma \quad \sigma_{\varphi_h} \circ *_h = \dagger \circ \sigma_{\varphi_h}, \quad (6.4)$$

and \triangleright_h as defined in formula (3.4) also satisfies (6.1). Under \dagger the RHS of relations (3.14), (3.15) are mapped into the RHS of relations (3.16), (3.17), provided

$$(\mathbf{a}^i)^{\dagger} = \mathbf{a}_i^+; \quad (6.5)$$

in this case we find, as requested

$$(A^i)^\dagger = A_i^+. \quad (6.6)$$

If $\mathfrak{g} = \mathfrak{sl}(N), \mathfrak{so}(N)$ and $\rho = \rho_d$ condition (6.5) is satisfied by choosing

$$v^{-1} = u = \begin{cases} \sqrt{y_{\mathfrak{sl}(N)}} & \text{if } \mathfrak{g} = \mathfrak{sl}(n) \\ \sqrt{y_{\mathfrak{so}(N)}} & \text{if } \mathfrak{g} = \mathfrak{so}(n) \end{cases} \quad (6.7)$$

$\mathcal{A}_{+, \mathfrak{so}(N), \rho_d}$ admits also an alternative $*$ -structure compatible with \triangleright_h , namely $(\tilde{A}_i^+)^\dagger = \tilde{A}_j^+ C^{ji}$ [24] together with a nonlinear equation for $(\tilde{A}^i)^\dagger_h$ [30] which we omit here; in this case one usually denotes the generators by X_i, ∂^i instead of $\tilde{A}_i^+, \tilde{A}^i$, because in the classical limit they become the Cartesian coordinates and partial derivatives of the N -dim Euclidean space respectively. The classical limit of this \dagger_h is

$$(a_i^+)^\dagger = a_j^+ c_{ji} \quad (a^i)^\dagger = -c_{ij} a^j; \quad (6.8)$$

using relations (6.8), (4.12), $\text{tr}(\rho_d) = 0$, $Sx = -x$ if $x \in \mathfrak{g}$, one finds again relations (6.4). \triangleright_h as defined in formula (3.4) also satisfies (6.1). Under \dagger the RHS of relation (3.14) is mapped into the RHS of relations (3.15), provided that $(\mathfrak{a}_i^+)^\dagger = \mathfrak{a}_j^+ c_{ji}$, *i.e.*

$$v = 1 \quad u = y_{\mathfrak{so}(N)}; \quad (6.9)$$

in this case we find, as requested

$$(A_i^+)^\dagger = A_j^+ C_{ji}, \quad (6.10)$$

and it is not difficult to show that $(A^i)^\dagger$ is the (nonlinear) function of A^i, A_i^+ which was found in Ref. [30].

7 Outlook, final remarks and conclusions

Given some solutions $\mathfrak{a}^i, \mathfrak{a}_i^+$ [in the form (3.9)] of equations (5.0.1-5.0.3), the A^i, A_i^+ defined through formulae (3.6) (where we choose a minimal \mathcal{F}) satisfy the deformed commutation relations of $\mathcal{A}_{\pm, \mathfrak{g}, \rho}^h$ and are covariant under the $U_h \mathfrak{g}$ action \triangleright_h defined in formula (3.4). The corresponding basic algebra homomorphism $f : \mathcal{A}_{\pm, \mathfrak{g}, \rho}^h \rightarrow \mathcal{A}_{\pm, \mathfrak{g}, \rho}[[h]]$ is defined iteratively starting from $f(\tilde{A}^i) := A^i$, $f(\tilde{A}_i^+) := A_i^+$. Explicit solutions $\mathfrak{a}^i, \mathfrak{a}_i^+$ of equations (5.0.1-5.0.3) are given by

- formulae (5.1.8), (5.1.9) for $\mathcal{A}_{\pm, \mathfrak{sl}(N), \rho_d}^h$;
- formulae (5.2.9), (5.2.12) for $\mathcal{A}_{+, \mathfrak{so}(N), \rho_d}^h$.

The main feature of such a realization (f, \triangleright_h) is that the \mathbf{g} -invariant ground state $|0\rangle$ as well as the first excited states $a_i^+|0\rangle$ of the classical Fock space representation are also respectively $U_h\mathbf{g}$ -invariant ground state $|0_h\rangle$ and first excited states $A_i^+|0_h\rangle$ of the deformed Fock space representation.

According to relation (1.1), all other elements of $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ satisfying the DCR of $\mathcal{A}_{\pm,\mathbf{g},\rho}^h$ can be written in the form

$$A^{\alpha i} = \alpha A^i \alpha^{-1} \quad A_{\alpha i}^+ = \alpha A_i^+ \alpha^{-1}, \quad (7.1)$$

with $\alpha = \mathbf{1}_{\mathcal{A}} + O(h) \in \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$. They are manifestly covariant under the $U_h\mathbf{g}$ -action $\triangleright_{h,\alpha}$ defined by

$$x \triangleright_{h,\alpha} a := \alpha \sigma_{\varphi_h}(x_{(\overline{1})}) a \sigma_{\varphi_h}(x_{(\overline{2})}) \alpha^{-1}. \quad (7.2)$$

For these realizations the deformed ground state in the Fock space representation reads $|0_h\rangle = \alpha|0\rangle$; thus in general the \mathbf{g} -invariant ground state and first excited states of the classical Fock space representation do not coincide with their deformed counterparts.

In this way we have found all possible pairs $(f_{\alpha}, \triangleright_{h,\alpha})$ making the diagram (1.2) in the introduction commutative.

Note that the change $\varphi_h \rightarrow \varphi_{h,v} = v \varphi_h(\cdot) v^{-1}$ [formula (2.1.22)] of the algebra isomorphism $U_h\mathbf{g} \rightarrow U\mathbf{g}[[h]]$ amounts to the particular transformation $(f, \triangleright_h) \rightarrow (f_{\alpha}, \triangleright_{h,\alpha})$, with $\alpha = \sigma(v)$.

In Sect 4 we have shown formula (4.5)] how to construct \mathbf{g} -invariants $I_h^n \in \mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$ in the form of homogeneous polynomials in A^i, A_i^+ . It is immediate to verify that under a transformation $(f, \triangleright_h) \rightarrow (f_{\alpha}, \triangleright_{h,\alpha})$ these I_h^n transform into $I_h^{\alpha n} := \alpha I_h^n \alpha^{-1}$.

In Sect. 6 we have shown (sticking to the explicit case of $\mathcal{A}_{\pm,sl(N),\rho_d}^h$ and $\mathcal{A}_{\pm,so(N),\rho_d}^h$) that, if $\mathcal{A}_{\pm,\mathbf{g},\rho}^h$ is a module $*$ -algebra [formula (6.1)] of the compact section of $U_h\mathbf{g}$ ($q > 1$), then one can choose (f, \triangleright_h) so that f is a $*$ -homomorphism, $f(b^{\dagger h}) = [f(b)]^{\dagger}$, and \triangleright_h also satisfies equation (6.1). It is straightforward to verify that $(f_{\alpha}, \triangleright_{h,\alpha})$ satisfy the same constraints provided that α is “unitary”, $\alpha^{\dagger} = \alpha^{-1}$.

Summing up, in the present work we have shown how to realize a deformed $U_h\mathbf{g}$ -covariant Weyl or Clifford algebra $\mathcal{A}_{\pm,\mathbf{g},\rho}^h$ within the undeformed one $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$. Given a deforming map f and a representation (π, V) of $\mathcal{A}_{\pm,\mathbf{g},\rho}$ on a vector space V , does $(\pi \circ f, V)$ provide also a representation of $\mathcal{A}_{\pm,\mathbf{g},\rho}^h$? In other words, can one interpret the elements of $\mathcal{A}_{\pm,\mathbf{g},\rho}^h$ as operators acting on V , if the elements of $\mathcal{A}_{\pm,\mathbf{g},\rho}$ are? If so, which specific role play the elements A^i, A_i^+ of $\mathcal{A}_{\pm,\mathbf{g},\rho}[[h]]$?

In view of the specific example we have examined in ref. [13] the answer to the first question seems to be always positive, whereas the converse statement is wrong: *e.g.* for any $h \in \mathbb{R}$ there are more (inequivalent) representations of the deformed algebra than representations of the undeformed one. This may seem a paradox, because in a h -formal-power sense f^{-1} can be defined and gives $f^{-1}(a^i) = \tilde{A}^i + O(h)$, $f^{-1}(a_i^+) = \tilde{A}_i^+ + O(h)$, whence at least for small h one would expect the deformed and undeformed representation theories to coincide; but in fact there is no warranty that, also in some operatorial sense, for small h the would-be $f^{-1}(a^i), f^{-1}(a_i^+)$ are ‘close’ to $\tilde{A}^i, \tilde{A}_i^+$. Of course, we are especially interested in Hilbert space representations of $*$ -algebras: in Ref. [13] we checked that in the operator-norm topology f^{-1} is ill-defined on all but exactly one deformed representation. Roughly speaking, the reason is that the ‘particle number’ observables n^i , which enter the transformation f (see *e.g.* (5.1.10)) are unbounded operators, therefore even for very small h the effect of the transformation on their large-eigenvalue eigenvectors can be so large to ‘push’ the latter out of the domain of definition of the operators in $f^{-1}(\mathcal{A}_{\pm, \mathbf{g}, \rho})$.

We are especially interested in the case of $*$ -algebras admitting Fock space representations. The results presented in the previous paragraphs could in principle be applied to models in quantum field theory or condensed matter physics by choosing representations ρ which are the direct sum of many copies of the same fundamental representation ρ_a ; this is what we have addressed in Ref. [22]. The different copies would correspond respectively to different space(time)-points or crystal sites.

One important physical issue is if $U_h \mathbf{g}$ -covariance necessarily implies exotic particle statistics. In view of what we have said the answer is no [31]. At least for compact \mathbf{g} and $U_h \mathbf{g}$ (h is real), the undeformed Fock space representation, which allows a ‘Bosons & Fermions’ particle interpretation, would carry also a representation of the deformed one. Next point is the role of the operators A^i, A_j^+ . Quadratic commutation‘rof the type (2.2.3-2.2.5) mean that A_i^+, A^i act as creators and annihilators of some excitations; a glance at (3.6), (7.1) shows that these are not the undeformed excitations, but some ‘composite’ ones. The last point is: what could these operators be good for. As an Hamiltonian H of the system we may choose a simple combination of the $U_h \mathbf{g}$ -invariants I_h^n of section 4; thus the Hamiltonian is $U_h \mathbf{g}$ -invariant and has a simple polynomial structure in the composite operators A^i, A_j^+ . H is also \mathbf{g} -invariant, but has a highly non-polynomial structure in the undeformed generators a^i, a_j^+ (it would be tempting to understand what kind of physics it could describe!). This suggests that the use of the A^i, A_j^+ instead of the a^i, a_j^+ should simplify the resolution of the corresponding dynamics (similarly to

what has been suggested in Ref. [32] for a 1-dim toy-model).

A Appendix

A.1 Proof of Lemma 1

Lemma 2 [13] *If $\mathcal{T} \in U\mathbf{g}[[h]]^{\otimes 3}$ is \mathbf{g} -invariant (i.e. $[\mathcal{T}, U\mathbf{g}[[h]]^{\otimes 3}] = 0$) then $m_{ij}S_i\mathcal{T}$, $m_{ij}S_j\mathcal{T}$ ($i, j = 1, 2, 3$, $i \neq j$) are \mathbf{g} -invariants belonging to $U\mathbf{g}[[h]]^{\otimes 2}$.*

(Here S_i denotes S acting on the i -th tensor factor, and m_{ij} multiplication of the i -th tensor factor by the j -th from the right.)

We may apply the previous lemma to $\mathcal{T} = \phi$, or $\mathcal{T} = \phi^{-1}$. Looking at the definition (2.1.9) one finds in particular the following \mathbf{g} -invariants

$$\begin{aligned} T_1 &:= m_{12}S_1\phi = (S\mathcal{F}^{(1)}\gamma \otimes \mathbf{1})\mathcal{F}(\mathcal{F}_{(1)}^{(2)} \otimes \mathcal{F}_{(2)}^{(2)}), \\ T_2 &:= m_{23}S_3\phi = (\mathcal{F}_{(1)}^{-1(1)} \otimes \mathcal{F}_{(2)}^{-1(1)})\mathcal{F}^{-1}(\mathbf{1} \otimes \gamma^{-1}S\mathcal{F}^{-1(2)}); \end{aligned} \quad (\text{A.1.1})$$

alternative expressions for these T_i can be obtained by applying the same operations to the identities

$$\phi q^{\frac{t_{12}+t_{13}}{2}} = q^{\frac{t_{12}}{2}}\mathcal{F}_{23,1}^{-1}\mathcal{F}_{12}^{-1}\mathcal{R}_{13}\mathcal{F}_{32}\mathcal{F}_{1,23}, \quad (\text{A.1.2})$$

$$q^{-\frac{t_{13}+t_{23}}{2}}\phi = \mathcal{F}_{12,3}^{-1}\mathcal{F}_{21}^{-1}\mathcal{R}_{13}^{-1}\mathcal{F}_{23}\mathcal{F}_{3,12}q^{-\frac{t_{23}}{2}}, \quad (\text{A.1.3})$$

which directly follow from relations (2.1.15), (2.1.9), (2.1.7) and the observation that $[\phi, q^{\frac{t_{12}+t_{13}+t_{23}}{2}}] = 0$. Applying $m_{12}S_1$ to (A.1.3), $m_{23}S_3$ to (A.1.2) we get

$$\begin{aligned} T_1 &= (S\mathcal{F}^{(2)}\gamma'^{-1} \otimes \mathbf{1})\mathcal{F}_{21}(\mathcal{F}_{(1)}^{(1)} \otimes \mathcal{F}_{(2)}^{(1)}), \\ T_2 &= (\mathcal{F}_{(1)}^{-1(2)} \otimes \mathcal{F}_{(2)}^{-1(2)})\mathcal{F}_{21}^{-1}(\mathbf{1} \otimes \gamma'S\mathcal{F}^{-1(1)}). \end{aligned} \quad (\text{A.1.4})$$

From eq. (A.1.1), (A.1.4) we easily find out that the inverse of T_i take the form

$$T_1^{-1} = \mathcal{F}^{-1}[\gamma^{-1}(S\mathcal{F}^{-1(1)})\mathcal{F}_{(1)}^{-1(2)} \otimes \mathcal{F}_{(2)}^{-1(2)}] \quad (\text{A.1.5})$$

$$= \mathcal{F}_{21}^{-1}[\gamma'(S\mathcal{F}^{-1(2)})\mathcal{F}_{(1)}^{-1(1)} \otimes \mathcal{F}_{(2)}^{-1(1)}], \quad (\text{A.1.6})$$

$$T_2^{-1} = [\mathcal{F}_{(1)}^{(1)} \otimes \mathcal{F}_{(2)}^{(1)}(S\mathcal{F}^{(2)})\gamma]\mathcal{F} \quad (\text{A.1.7})$$

$$= [\mathcal{F}_{(1)}^{(2)} \otimes \mathcal{F}_{(2)}^{(2)}(S\mathcal{F}^{(1)})\gamma'^{-1}]\mathcal{F}_{21}, \quad (\text{A.1.8})$$

since $[T_i, \mathcal{F}_{(1)}^{\pm 1(j)} \otimes \mathcal{F}_{(2)}^{\pm 1(j)}] = 0$, with $i, j = 1, 2$.

If \mathcal{F} is minimal then it is easy to verify that, according to (2.1.16) and their definitions (A.1.1), $T_i \equiv \mathbf{1}^{\otimes 3}$. In the latter case the last four relations are equivalent to relations (3.10-3.13).

A.2 Proof of Proposition 5

We start by expressing $(A...A)^{I_n} \equiv A^{i_1}...A^{i_{h_n}}$, $(A^+...A^+)_{J_n} \equiv A_{j_1}^+...A_{j_{k_n}}^+$ respectively in the form $(a...a)\sigma(\cdot)$, $(a^+...a^+)\sigma(\cdot)$. First note that

$$\begin{aligned} A^{i_1} A^{i_2} &\stackrel{(3.17)}{=} \rho(\gamma^{-1} S \mathcal{F}^{-1(1)})_{l_1}^{i_1} \rho(\gamma^{-1} S \mathcal{F}^{-1(1')})_{l_2}^{i_2} a^{l_1} \sigma(\mathcal{F}^{-1(2)}) a^{l_2} \sigma(\mathcal{F}^{-1(2')}) \\ &= \rho^\vee(\mathcal{F}^{-1(1)} \gamma')_{i_1}^{l_1} \rho^\vee(\mathcal{F}^{-1(1')} \gamma')_{i_2}^{l_2} a^{l_1} \sigma(\mathcal{F}^{-1(2)}) a^{l_2} \sigma(\mathcal{F}^{-1(2')}) \\ &\stackrel{(3.5), (2.2.2)}{=} \rho^\vee(\mathcal{F}^{-1(1)} \gamma')_{i_1}^{l_1} \rho^\vee(\mathcal{F}_{(1)}^{-1(2)} \mathcal{F}^{-1(1')} \gamma')_{i_2}^{l_2} a^{l_1} a^{l_2} \sigma(\mathcal{F}_{(2)}^{-1(2)} \mathcal{F}^{-1(2')}) \\ &= a^{l_1} a^{l_2} \left[\left(\rho_{i_1}^{\vee l_1} \otimes \rho_{i_2}^{\vee l_2} \otimes \sigma \right) \left(\mathcal{F}_{1,23}^{-1} \mathcal{F}_{23}^{-1} (\gamma'^{\otimes 2} \otimes \mathbf{1}) \right) \right] \end{aligned}$$

whence, by repeated application, we find

$$(A...A)^{I_n} \stackrel{(4.9)}{=} (a...a)^{L_n} \left[\left((\rho^{\vee \otimes h_n})_{I_n}^{L_n} \otimes \sigma \right) \left(\mathcal{F}_{12...(h_n+1)}^{-1} (\gamma'^{\otimes h_n} \otimes \mathbf{1}) \right) \right];$$

similarly, starting from relation (3.14) we find

$$(A^+...A^+)_{J_n} = (a^+...a^+)_{M_n} \left[\left((\rho^{\otimes k_n})_{J_n}^{M_n} \otimes \sigma \right) \left(\mathcal{F}_{12...(k_n+1)}^{-1} \right) \right].$$

Putting these results together we find

$$\begin{aligned} &(A^+...A^+)_{J_n} (A...A)^{I_n} \stackrel{(2.2.2), (3.5)}{=} \\ &(a^+...a^+)_{M_n} (a...a)^{L_n} \left[\left((\rho^{\otimes k_n})_{J_n}^{M_n} \otimes (\rho^{\vee \otimes h_n})_{I_n}^{L_n} \otimes \sigma \right) \left(\mathcal{F}_{12...(b_n+1)}^{-1} (\mathbf{1}^{\otimes k_n} \otimes \gamma'^{\otimes h_n} \otimes \mathbf{1}) \right) \right], \end{aligned}$$

whence,

$$I_h^n \stackrel{(4.8)}{\propto} (a^+...a^+)_{M_n} (a...a)^{L_n} \left[\left((\rho^{\otimes k_n})_{J_n}^{M_n} \otimes (\rho^{\vee \otimes h_n})_{I_n}^{L_n} \otimes \sigma \right) \mathcal{F}_{12...(b_n+1)}^{-1} \mathcal{F}_{12...b_n, b_n+1} \right] d_{M_n}^{L_n}. \quad (\text{A.2.1})$$

We prove now that

$$\mathcal{F}_{12...(b+1)}^{-1} \mathcal{F}_{12...b} = \phi_{(b-1)b(b+1)}^{-1} \phi_{(b-2),(b-1)b,(b+1)}^{-1} \cdots \phi_{1,2...b,(b+1)}^{-1} \mathcal{F}_{12...m,m+1}^{-1}; \quad (\text{A.2.2})$$

then the claim will follow from relation (A.2.1) and the observation that

$$\left[\left(\rho^{\otimes k_n} \otimes \rho^{\vee \otimes h_n} \otimes \text{id} \right) \left(\mathcal{F}_{12...b_n, b_n+1} \right) \right]_{J_n I_n}^{M_n L_n} d_{M_n}^{L_n} \stackrel{(4.6)}{=} \varepsilon(\mathcal{F}^{(1)}) \mathcal{F}^{(2)} d_{J_n}^{I_n} \stackrel{(2.1.1)}{=} d_{J_n}^{I_n}. \quad (\text{A.2.3})$$

To prove relation (A.2.2) we start from

$$\phi_{123}^{-1} \mathcal{F}_{12,3}^{-1} \stackrel{(2.1.9)}{=} \mathcal{F}_{1,23}^{-1} \mathcal{F}_{23}^{-1} \mathcal{F}_{12} \stackrel{(4.9)}{=} \mathcal{F}_{123}^{-1} \mathcal{F}_{12};$$

this is relation (A.2.2) for $b = 2$. Applying $\text{id} \otimes \Delta \otimes \text{id}$ and multiplying the result from the left by ϕ_{234}^{-1} we find

$$\begin{aligned} \phi_{234}^{-1} \phi_{1,23,4}^{-1} \mathcal{F}_{123,4}^{-1} &= \phi_{234}^{-1} \mathcal{F}_{1,234}^{-1} \mathcal{F}_{2,34}^{-1} \mathcal{F}_{1,23} \\ &\stackrel{(2.1.10)}{=} \mathcal{F}_{1,234}^{-1} \phi_{234}^{-1} \mathcal{F}_{2,34}^{-1} \mathcal{F}_{1,23} \\ &\stackrel{(2.1.9)}{=} \mathcal{F}_{1,234}^{-1} \mathcal{F}_{2,34}^{-1} \mathcal{F}_{34}^{-1} \mathcal{F}_{23} \mathcal{F}_{1,23}, \\ &\stackrel{(4.9)}{=} \mathcal{F}_{1233}^{-1} \mathcal{F}_{123}, \end{aligned}$$

i.e. relation (A.2.2) for $b = 3$. Applying to the latter relation $\text{id} \otimes \Delta^{(2)} \otimes \text{id}$ and multiplying the result from the left by ϕ_{345}^{-1} we find relation (A.2.2) for $b = 4$, and so on \square .

A.3 Proof of Proposition 6

$$\begin{aligned}
0 &\stackrel{(2.2.3)}{=} A^i A^j \mp P_{hk}^{Fji} A^k A^h \\
&\stackrel{(3.16)}{=} (\mathbf{1} \mp P^F)_{hk}^{ji} \rho(\mathcal{F}^{(1)})_l^h \rho(\mathcal{F}^{(1')})_m^k \sigma(\mathcal{F}^{(2)}) \mathbf{a}^m \sigma(\mathcal{F}^{(2')}) \mathbf{a}^l \\
&\stackrel{(3.5), (2.2.2)}{=} (\mathbf{1} \mp P^F)_{hk}^{ji} \rho(\mathcal{F}^{(1)} \mathcal{F}_{(2')}^{(2')})_m^k \rho(\mathcal{F}^{(1)})_l^h \sigma(\mathcal{F}_{(1)}^{(2)} \mathcal{F}_{(1')}^{(2')}) \mathbf{a}^m \mathbf{a}^l;
\end{aligned}$$

multiplying both sides from the left by $(\rho \otimes \rho \otimes \sigma)(\mathcal{F}_{1,23}^{-1} \mathcal{F}_{23}^{-1})$ and noting that

$$P_{12}^F \stackrel{(2.2.11), (2.2.19)}{=} [(\rho \otimes \rho \otimes \sigma) \mathcal{F}_{12} \mathcal{F}_{12,3}] U_{12} [(\rho \otimes \rho \otimes \sigma) \mathcal{F}_{12,3}^{-1} \mathcal{F}_{12}^{-1}],$$

we find

$$\{\mathbf{1}^{\otimes 3} \mp [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}_{1,23}^{-1} \mathcal{F}_{23}^{-1} \mathcal{F}_{12} \mathcal{F}_{12,3}] U_{12} [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}_{12,3}^{-1} \mathcal{F}_{12}^{-1} \mathcal{F}_{23} \mathcal{F}_{1,23}]\}^{hk} \mathbf{a}^m \mathbf{a}^l = 0,$$

i.e. relation (5.0.1), once we take definitions (5.0.4), (2.1.9) into account. Using definition (3.14) one can prove in a similar way that relations (2.2.4), (5.0.2) are equivalent. Similarly,

$$\begin{aligned}
0 &\stackrel{(2.2.5)}{=} A^i A_j^+ - \delta_j^i \mathbf{1}_{\mathcal{A}} \mp \tilde{P}_{jk}^F {}^{ih} A_h^+ A^k \\
&\stackrel{(3.16), (3.14)}{=} \rho(\mathcal{F}^{(1)})_l^i \sigma(\mathcal{F}^{(2)}) \mathbf{a}^l \mathbf{a}_m^+ \sigma(\mathcal{F}^{-1(2')}) \rho(\mathcal{F}^{-1(1')})_j^m \\
&\quad - \delta_j^i \mathbf{1}_{\mathcal{A}} \mp \mathbf{a}_m^+ \sigma(\mathcal{F}^{-1(2)}) \rho(\mathcal{F}^{-1(1)})_h^m \tilde{P}_{jk}^F {}^{ih} \rho(\mathcal{F}^{(1')})_l^k \sigma(\mathcal{F}^{(2)}) \mathbf{a}^l;
\end{aligned}$$

multiplying both sides by $\rho(\mathcal{F}^{-1(1)})_i^{i'} \sigma(\mathcal{F}^{-1(2)})$ from the left and by $\rho(\mathcal{F}^{(1')})_{j'}^j \sigma(\mathcal{F}^{(2')})$ from the right, and noting that

$$\tilde{P}_{12}^F \stackrel{(2.2.12), (2.2.19)}{=} [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}_{12} \mathcal{F}_{12,3}] V_{12} [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}_{12,3}^{-1} \mathcal{F}_{12}^{-1}],$$

we get

$$\begin{aligned}
0 &= \mathbf{a}^{i'} \mathbf{a}_{j'}^+ - \delta_{j'}^{i'} \mathbf{1}_{\mathcal{A}} \mp \rho(\mathcal{F}^{-1(1)})_i^{i'} \sigma(\mathcal{F}^{-1(2)}) \mathbf{a}_m^+ \left\{ [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}_{13}^{-1} \mathcal{F}_{12} \mathcal{F}_{12,3}] \right. \\
&\quad \left. \times V_{12} [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}_{12,3}^{-1} \mathcal{F}_{12}^{-1} \mathcal{F}_{13}] \right\}_{jl}^{im} \mathbf{a}^l \rho(\mathcal{F}^{(1')})_{j'}^j \sigma(\mathcal{F}^{(2')}) \\
&\stackrel{(3.5), (2.2.2)}{=} \mathbf{a}^{i'} \mathbf{a}_{j'}^+ - \delta_{j'}^{i'} \mathbf{1}_{\mathcal{A}} \mp \mathbf{a}_m^+ \left\{ [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}_{1,23}^{-1} \mathcal{F}_{13}^{-1} \mathcal{F}_{12} \mathcal{F}_{12,3}] \right. \\
&\quad \left. \times V_{12} [(\rho^{\otimes 2} \otimes \sigma) \mathcal{F}_{12,3}^{-1} \mathcal{F}_{12}^{-1} \mathcal{F}_{13} \mathcal{F}_{1,23}] \right\}_{j'l}^{i'm} \mathbf{a}^l,
\end{aligned}$$

whence the equivalence between relations (2.2.5), (5.0.3) follows, once one recalls the definition (2.1.9). \square

A.4 Some properties of special and q-special functions

The following results can be found in standard textbooks. If the parameters $a, b, c \in \mathbf{C}$ are such that none of the quantities $c-1, a-b, a+b-c$ is a positive integer, the general solution of the hypergeometric differential equation in the complex z -plane

$$y''(1-z)z + y'[c - (a+b+1)z] - yab = 0 \quad (\text{A.4.1})$$

can be expressed as some combinations

$$y(z) = \alpha F(a, b, c; z) + \beta z^{1-c} F(1+a-c, 1+b-c, 2-c; z), \quad (\text{A.4.2})$$

$$= \gamma F(a, b, a+b+1-c; 1-z) + \delta (1-z)^{c-a-b} F(c-a, c-b, c+1-a-b; 1-z), \quad (\text{A.4.3})$$

where $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ and $F(a, b, c; z)$ is the hypergeometric function. As known,

$$F(a, b, c; 0) = 1, \quad (\text{A.4.4})$$

$$\frac{d}{dz} F(a, b, c; z) = \frac{ab}{c} F(a+1, b+1, c+1; z). \quad (\text{A.4.5})$$

The combinations (A.4.2), (A.4.3) explicitly display the singular and non-singular part of the solution respectively around the poles $x = 0, 1$. An essential identity to determine the asymptotic behaviour of a solution y around the pole $x = 0$ (resp. $x = 1$), known its asymptotic behaviour around the pole $x = 1$ (resp. $x = 0$), is

$$\begin{aligned} F(a, b, c; z) &= \frac{B(c, c-a-b)}{B(c-a, c-b)} F(a, b, a+b+1-c; 1-z) \\ &+ \frac{B(c, a+b-c)}{B(a, b)} (1-z)^{c-a-b} F(c-a, c-b, c+1-a-b; 1-z); \end{aligned} \quad (\text{A.4.6})$$

Here $\Gamma(a)$ and $B(a, b)$ are Euler's Γ - and β -functions respectively; as known,

$$\Gamma(a+1) = a\Gamma(a) \quad B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (\text{A.4.7})$$

A less obvious property is

$$\Gamma(a)\Gamma(-a) = -\frac{\pi}{a \sin \pi a}. \quad (\text{A.4.8})$$

The q -gamma function Γ_q can be defined when $|q| < 1$ by [33]

$$\Gamma_q(a) := (1 - q^{1-a}) \prod_{k=0}^{\infty} \frac{(1 - q^{k+1})}{(1 - q^{a+k})} = (1 - q^{1-a}) \sum_{n=0}^{\infty} \frac{(q^{1-a}; q)_n}{(q^a; q)_n} q^{na}, \quad (\text{A.4.9})$$

where $(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k)$; it satisfies the following modified version of the property (A.4.7)₁:

$$\Gamma_q(a+1) = (a)_q \Gamma_q(a), \quad (a)_q := \frac{(q^a - 1)}{(q - 1)}. \quad (\text{A.4.10})$$

We introduce also a different version of the q -gamma function by

$$\tilde{\Gamma}_q(a) := \Gamma_{q^2}(a) q^{-\frac{a(a-3)}{2}}; \quad (\text{A.4.11})$$

the latter satisfies

$$\tilde{\Gamma}_q(a+1) = [a]_q \tilde{\Gamma}_q(a), \quad [a]_q := \frac{(q^a - q^{-a})}{(q - q^{-1})}. \quad (\text{A.4.12})$$

A.5 Proof of Theorem 1

Proof. We need to show that equations (5.0.1-5.0.3) are fulfilled. We get rid of indices by introducing the following vector notation:

$$\begin{aligned} (aa)^{ij} &:= a^i a^j & (a^+ a^+)_{ij} &:= a_i^+ a_j^+ \\ (av)^{ij} &:= a^i v^j & (va)^{ij} &:= v^i a^j \\ (a^+ w)_{ij} &:= a_i^+ w_j & (wa^+)_{ij} &:= w_i a_j^+ \\ w \cdot a &:= w_i a^i & a^+ \cdot v &:= a_i^+ v^i, \end{aligned}$$

where $v \equiv (v^i) \in \mathbf{C}^N$, $w \equiv (w_i) \in \mathbf{C}^N$ denote arbitrary covariant and contravariant vectors respectively. If we plug (5.1.7) into (5.0.1-5.0.3), factor out of (5.0.1) and (5.0.2) $I(n) I(n+1)$ and $\tilde{I}(n) \tilde{I}(n+1)$ respectively, multiply eq. (5.0.3) by $v^j w_i$, then we find the equivalent system (in vector notation)

$$aa = \pm (M^{-1} P M) aa \quad (\text{A.5.1})$$

$$a^+ a^+ = \pm a^+ a^+ (M^{-1} P M) \quad (\text{A.5.2})$$

$$\frac{(n+1)_{q^{\pm 2}}}{(n+1)} (w \cdot a) \cdot (a^+ \cdot v) = w \cdot v \mathbf{1}_{\mathcal{A}} \pm q^{\pm 1} \frac{n_{q^{\pm 2}}}{n} wa^+ (M^{-1} V M) va. \quad (\text{A.5.3})$$

It is straightforward to show that

$$\begin{aligned} A va &= \pm (n-1) va & a^+ w A &= -wa^+ \mp a^+ w + (w \cdot a) a^+ a^+ \\ wa^+ P &= a^+ w & a^+ w P &= wa^+. \end{aligned} \quad (\text{A.5.4})$$

As a consequence one finds, in particular,

$$\begin{aligned} a^+ w (A+P)^k &= (\mp 1)^k a^+ w \pm \frac{(\pm n)^k - (\mp 1)^k}{n+1} (w \cdot a) a^+ a^+ \quad \Rightarrow \\ \Rightarrow a^+ w q^{A+P} &= q^{\pm 1} a^+ w \pm \frac{q^{\mp n} - q^{\mp 1}}{n+1} (w \cdot a) a^+ a^+ \end{aligned} \quad (\text{A.5.5})$$

Let us prove eq. (A.5.1), (A.5.2). The matrix M (5.0.4) takes the form

$$M \stackrel{(2.1.21)}{=} \lim_{x_0, y_0 \rightarrow 0^+} \left\{ x_0^{-2\hbar P} \vec{P} \exp \left[-2\hbar \int_{x_0}^{1-y_0} dx \left(\frac{P}{x} + \frac{A}{x-1} \right) \right] y_0^{2\hbar A} \right\}; \quad (\text{A.5.6})$$

the contributions of the central terms $-2\frac{\mathbf{1}^{\otimes 3}}{N}$, $-\frac{2}{N}\mathbf{1}^{\otimes 3} \otimes n$ to the integral are cancelled by the corresponding contributions from $x_0^{-\hbar t_{12}}$, $y_0^{\hbar t_{23}}$, in the limit $x_0, y_0 \rightarrow 0^+$. Since aa is an ‘eigenvector’ both of A and P , the path-order \vec{P} becomes redundant and we find that M acts trivially on M :

$$\begin{aligned} Maa &= aa \lim_{x_0, y_0 \rightarrow 0^+} \left\{ x_0^{\mp 2\hbar} \exp \left[-2\hbar \int_{x_0}^{1-y_0} dx \left(\pm \frac{1}{x} \pm \frac{n-2}{x-1} \right) \right] y_0^{\pm 2\hbar(n-2)} \right\} \\ &= aa \lim_{x_0, y_0 \rightarrow 0^+} (1-y_0)^{\mp 2\hbar} (1-x_0)^{\pm 2\hbar(n-2)} = aa. \end{aligned} \quad (\text{A.5.7})$$

Therefore $MPMaa = \pm aa$, Q.E.D. Similarly one proves eq. (A.5.2).

In order to prove eq. (A.5.3) it is convenient to recast $M^{-1}VM$ in a more manageable form. Permuting the second and third tensor factor in eq. (2.1.15)₍₁₎, we find

$$\phi_{213}^{-1} q^{\frac{t_{12}}{2}} \phi_{123} = q^{\frac{t_{12}+t_{13}}{2}} \phi_{132} q^{-\frac{t_{23}}{2}}, \quad (\text{A.5.8})$$

whence

$$M^{-1}VM = PM_{21}^{-1} q^{\pm 1 + \frac{1}{N} + \rho_d \otimes^2 (\frac{t}{2})} M^{(2.1.19), (5.1.6)} Pq^{A+P} \lim_{x \rightarrow 0^+} x^{-2B} f(x) q^{-A}, \quad (\text{A.5.9})$$

where f is the $\sigma(Usl(N))[[h]]$ -valued $N^2 \times N^2$ matrix satisfying the differential equation and asymptotic conditions

$$f' = 2\hbar \left(\frac{B}{x} + \frac{A}{x-1} \right) f \quad \lim_{x \rightarrow 1} f(x)(1-x)^{-2A} = 1 \quad (\text{A.5.10})$$

[the latter are obtained from eq. (2.1.17) by permuting the second and third tensor factor and by getting rid of the central terms involved in $(\rho \otimes \rho \otimes \sigma)(t_{ij})$ (formulae (5.1.6)) since, as in formula (A.5.6), the latter cancel with each other in the limit $x \rightarrow 0$].

It is convenient to introduce in $\mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]]$ a grading g , by setting $g(b) = l \in \mathbf{Z}$ iff $[n, b] = lb$, $b \in \mathcal{A}_{\pm, \mathbf{g}, \rho}[[h]]$. Since $g(fva) = -1$, and $f v^i a^j$ is a doubly contravariant tensor, its most general expansion is

$$f(x)va = avf_1(x) + vaf_2(x) + aa(a^+ \cdot v)f_3(x), \quad (\text{A.5.11})$$

where f_i are invariants with $g(f_i) = 0$; therefore $f_i = f_i(n)$. Thus we find

$$\begin{aligned} & wa^+ \cdot (M^{-1}VM)va \\ & \stackrel{(\text{A.5.4}), (\text{A.5.9})}{=} a^+ w q^{A+P} \lim_{x \rightarrow 0^+} x^{-2\hbar B} f(x) q^{-A} va \\ & \stackrel{(\text{A.5.4}), (\text{A.5.5})}{=} \lim_{x \rightarrow 0^+} x^{\mp 2\hbar(n-1)} \left[q^{\mp 1} a^+ w \pm \frac{q^{\pm n} - q^{\mp 1}}{n+1} (w \cdot a) a^+ a^+ \right] \cdot (f(x)va) q^{\mp(n-1)} \end{aligned}$$

$$\begin{aligned}
& \stackrel{(A.5.11)}{=} q^{\mp(n-1)} \lim_{x \rightarrow 0^+} x^{\mp 2\hbar(n-1)} \left[q^{\mp 1} a^+ w \pm \frac{q^{\pm n} - q^{\mp 1}}{n+1} (w \cdot a) a^+ a^+ \right] \times \\
& \quad \left[a v f_1(x) + v a f_2(x) + a a (a^+ \cdot v) f_3(x) \right] \\
& = q^{\mp(n-1)} \lim_{x \rightarrow 0^+} x^{\mp 2\hbar(n-1)} \left\{ q^{\mp 1} (w \cdot v) (n f_1 \mp f_2) + (w \cdot a) (a^+ \cdot v) \left[q^{\mp 1} (n f_3 \pm f_2) \right. \right. \\
& \quad \left. \left. + n \frac{q^{\pm n} - q^{\mp 1}}{n+1} (f_1 \pm f_2 + (n+1) f_3) \right] \right\} \\
& = q^{\mp n} \left\{ (w \cdot v) l_1 + (w \cdot a) (a^+ \cdot v) \left[l_2 + n \frac{q^{\pm(n+1)} - 1}{n+1} l_3 \right] \right\}, \tag{A.5.12}
\end{aligned}$$

where we have defined

$$\begin{aligned}
l_1 &:= \lim_{x \rightarrow 0^+} x^{\mp 2\hbar(n-1)} (n f_1(x) \mp f_2(x)) \\
l_2 &:= \lim_{x \rightarrow 0^+} x^{\mp 2\hbar(n-1)} (n f_3(x) \pm f_2(x)) \\
l_3 &:= \lim_{x \rightarrow 0^+} x^{\mp 2\hbar(n-1)} [f_1(x) \pm f_2(x) + (n+1) f_3(x)]. \tag{A.5.13}
\end{aligned}$$

To evaluate the limits l_i let us consider the linear system of first order differential equations satisfied by f_i . From (A.5.10) we find

$$f'_1 = \hbar \left[\pm \left(\frac{1}{1-x} + \frac{n-1}{x} \right) f_1 - \frac{f_2}{x} \right] \tag{A.5.14}$$

$$f'_2 = \hbar \left[\frac{f_1}{1-x} \mp \left(\frac{n-1}{1-x} + \frac{1}{x} \right) f_2 \right] \tag{A.5.15}$$

$$f'_3 = \hbar \left[\mp \frac{f_1}{1-x} + \frac{f_2}{x} \mp \left(\frac{1}{1-x} - \frac{1}{x} \right) (n-1) f_3 \right] \tag{A.5.16}$$

and the asymptotic condition

$$\lim_{x \rightarrow 1} f_1(x) = 0 = \lim_{x \rightarrow 1} f_3(x) \quad \lim_{x \rightarrow 1} f_2(x) (1-x)^{\mp 2\hbar(n-1)} = 1. \tag{A.5.17}$$

The first two equations can be solved separately; then the third will yield f_3 in terms of f_1, f_2 just by an integration. Actually one of the combination we are interested in, $[f_1(x) \pm f_2(x) + (n+1) f_3(x)]$, satisfies a completely decoupled equation, $\frac{d}{dx} [f_1(x) \pm f_2(x) + (n+1) f_3(x)] = \pm 2\hbar(n-1) \left[\frac{1}{x} + \frac{1}{x-1} \right] [f_1(x) \pm f_2(x) + (n+1) f_3(x)]$, which [taking conditions (A.5.17) into account] is easily integrated to

$$f_1(x) \pm f_2(x) + (n+1) f_3(x) = \pm [x(1-x)]^{\pm 2\hbar(n-1)}. \tag{A.5.18}$$

This will yield therefore f_3 in terms of f_1, f_2 . Dividing (A.5.14) by f_1 , (A.5.15) by f_2 we find

$$\frac{f'_1}{f_1} = 2\hbar \left[\pm \left(\frac{1}{1-x} + \frac{n-1}{x} \right) - \frac{1}{x} \frac{f_2}{f_1} \right] \tag{A.5.19}$$

$$\frac{f'_2}{f_2} = 2\hbar \left[\frac{1}{(1-x)} \frac{f_1}{f_2} \mp \left(\frac{n-1}{1-x} + \frac{1}{x} \right) \right] \tag{A.5.20}$$

taking the difference of the two, one finds a Riccati equation in the unknown $u := \frac{f_1}{f_2}$:

$$\frac{u'}{u} = \frac{d}{dx} \ln\left(\frac{f_1}{f_2}\right) = \frac{f_1'}{f_1} - \frac{f_2'}{f_2} = 2\hbar \left[\pm n \left(\frac{1}{x} + \frac{1}{1-x} \right) - \frac{u^{-1}}{x} - \frac{u}{1-x} \right]; \quad (\text{A.5.21})$$

this should be supplemented with the condition $u \xrightarrow{x \rightarrow 1} 0$. To get rid of its nonlinearity one can transform it into a (linear) second order equation in an unknown $y(x)$ by a standard substitution, which in this case takes the form

$$u = \frac{y'(1-x)}{y \cdot 2\hbar}; \quad (\text{A.5.22})$$

the new equation will read

$$y''(1-x)x - y'(x \pm n2\hbar) + (2\hbar)^2 y = 0. \quad (\text{A.5.23})$$

We recognize the hypergeometric equation [formula (A.4.1)] with parameters

$$a = \pm 2\hbar \quad b = \mp 2\hbar \quad c = \mp 2n\hbar. \quad (\text{A.5.24})$$

Its general solution can be expressed in the form (A.4.3), in terms of the hypergeometric function F . Imposing the condition $\lim_{x \rightarrow 1} \frac{y'(1-x)}{2\hbar y} = 0$ one finds that it must be $\delta = 0$, implying

$$\frac{f_1}{f_2} = u = \frac{1-x}{2\hbar} \frac{d}{dx} \ln [F(\pm 2\hbar, \mp 2\hbar, 1 \pm 2\hbar n; 1-x)]. \quad (\text{A.5.25})$$

We can now replace this result in the RHS in eq. (A.5.20):

$$\frac{d}{dx} \ln(f_2) = \frac{d}{dx} \ln [F(\pm 2\hbar, \mp 2\hbar, 1 \pm 2\hbar n; 1-x)] \mp 2\hbar \left(\frac{n-1}{1-x} + \frac{1}{x} \right); \quad (\text{A.5.26})$$

taking into account the condition (A.5.17), the latter is integrated to

$$f_2(x) = x^{\mp 2\hbar} (1-x)^{\pm 2\hbar(n-1)} F(\pm 2\hbar, \mp 2\hbar, 1 \pm 2\hbar n; 1-x). \quad (\text{A.5.27})$$

Finally, we find

$$f_1(x) = u(x)f_2(x) = -\frac{1}{2\hbar} F'(\pm 2\hbar, \mp 2\hbar, 1 \pm 2\hbar n; 1-x) x^{\mp 2\hbar} (1-x)^{1 \pm 2\hbar(n-1)}. \quad (\text{A.5.28})$$

From properties (A.4.4), (A.4.5) we can easily read off the asymptotic behaviour of f_2, f_1 for $x \rightarrow 0^+$:

$$f_2(x) \stackrel{(\text{A.4.6})}{=} x^{\mp 2\hbar} (1-x)^{\pm 2\hbar(n-1)} \left[\frac{B(1 \pm 2\hbar n, 1 \pm 2\hbar n)}{B(1 \pm 2\hbar(n+1), 1 \pm 2\hbar(n-1))} F(\pm 2\hbar, \mp 2\hbar, \mp 2\hbar n; x) + x^{1 \pm 2\hbar n} \times \right. \\ \left. \frac{B(1 \pm 2\hbar n, -1 \mp 2\hbar n)}{B(\pm 2\hbar, \mp 2\hbar)} F(1 \pm 2(n+1)\hbar, 1 \mp 2\hbar(n-1), 2 \pm 2\hbar n; x) \right] \stackrel{(\text{A.4.4}), (\text{A.4.7})}{\approx} x^{\mp 2\hbar} \frac{\Gamma(1 \pm 2\hbar n) \Gamma(1 \pm 2\hbar n)}{\Gamma(1 \pm 2\hbar(n+1)) \Gamma(1 \pm 2\hbar(n-1))}$$

$$\begin{aligned}
f_1(x) &\stackrel{(A.4.5)}{=} \frac{2\hbar}{1\pm 2\hbar n} F(1\pm 2\hbar, 1\mp 2\hbar, 2\pm 2\hbar n; 1-x) x^{\mp 2\hbar} (1-x)^{1\pm 2\hbar(n-1)} \\
&\stackrel{(A.4.6)}{=} \frac{2\hbar}{1\pm 2\hbar n} x^{\mp 2\hbar} (1-x)^{1\pm 2\hbar(n-1)} \left[\frac{B(2\pm 2\hbar n, \pm 2\hbar n)}{B(1\pm 2\hbar(n+1), 1\pm 2\hbar(n-1))} F(1\pm 2\hbar, 1\mp 2\hbar, 1\mp 2\hbar n; x) \right. \\
&\quad \left. + x^{\pm 2\hbar n} \frac{B(2\pm 2\hbar n, \mp 2\hbar n)}{B(1\pm 2\hbar, 1\mp 2\hbar)} F(1\pm 2(n+1)\hbar, 1\mp 2\hbar(n-1), 1\pm 2\hbar n; x) \right] \\
&\stackrel{(A.4.4)}{\approx} \frac{2\hbar}{1\pm 2\hbar n} x^{\mp 2\hbar} \left[\frac{B(2\pm 2\hbar n, \pm 2\hbar n)}{B(1\pm 2\hbar(n+1), 1\pm 2\hbar(n-1))} + x^{\pm 2\hbar n} \frac{B(2\pm 2\hbar n, \mp 2\hbar n)}{B(1\pm 2\hbar, 1\mp 2\hbar)} \right] \\
&\stackrel{(A.4.7)}{=} x^{\mp 2\hbar} \left[\pm \frac{1}{n} \frac{\Gamma(1\pm 2\hbar n)\Gamma(1\pm 2\hbar n)}{\Gamma(1\pm 2\hbar(n+1))\Gamma(1\pm 2\hbar(n-1))} + 2\hbar x^{\pm 2\hbar n} \frac{\Gamma(1\pm 2\hbar n)\Gamma(\mp 2\hbar n)}{\Gamma(1\pm 2\hbar)\Gamma(1\mp 2\hbar)} \right].
\end{aligned}$$

For the combination $nf_1 \mp f_2$ we thus find

$$\begin{aligned}
nf_1 \mp f_2 &\stackrel{x \rightarrow 0}{\approx} 2\hbar x^{\pm 2\hbar(n-1)} \frac{\Gamma(1\pm 2\hbar n)\Gamma(\mp 2\hbar n)}{\Gamma(1\pm 2\hbar)\Gamma(1\mp 2\hbar)} \stackrel{(A.4.7)}{=} \mp nx^{\pm 2\hbar(n-1)} \frac{\Gamma(\pm 2\hbar n)\Gamma(\mp 2\hbar n)}{\Gamma(\pm 2\hbar)\Gamma(\mp 2\hbar)} \\
&\stackrel{(A.4.8)}{=} \mp \frac{q - q^{-1}}{q^n - q^{-n}} x^{\pm 2\hbar(n-1)} = \mp \frac{1}{[n]_q} x^{\pm 2\hbar(n-1)}.
\end{aligned}$$

The limits l_i are thus given by

$$\begin{aligned}
l_1 &= \mp \frac{1}{[n]_q} \\
l_3 &= \pm 1 \\
l_2 &= \frac{n}{n+1} l_3 - \frac{1}{n+1} l_1 = \pm \frac{n}{n+1} \left(1 + \frac{1}{[n]_q} \right),
\end{aligned} \tag{A.5.29}$$

which plugged into eq. (A.5.12) give

$$\begin{aligned}
wa^+ \cdot (M^{-1} V M) va &= q^{\mp n} \left[-\frac{(w \cdot v)}{[n]_q} + (w \cdot a)(a^+ \cdot v) \frac{n}{n+1} \left(\frac{1}{[n]_q} + q^{\pm(n+1)} \right) \right] \\
&= \mp \frac{q^{\mp 1}}{n_{q^{\pm 2}}} \pm (w \cdot a)(a^+ \cdot v) q^{\mp 1} \frac{(n+1)_{q^{\pm 2}}}{(n+1)} \frac{n}{n_{q^{\pm 2}}};
\end{aligned} \tag{A.5.30}$$

eq. (A.5.3) is manifestly satisfied once we replace the latter result in it. \square

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